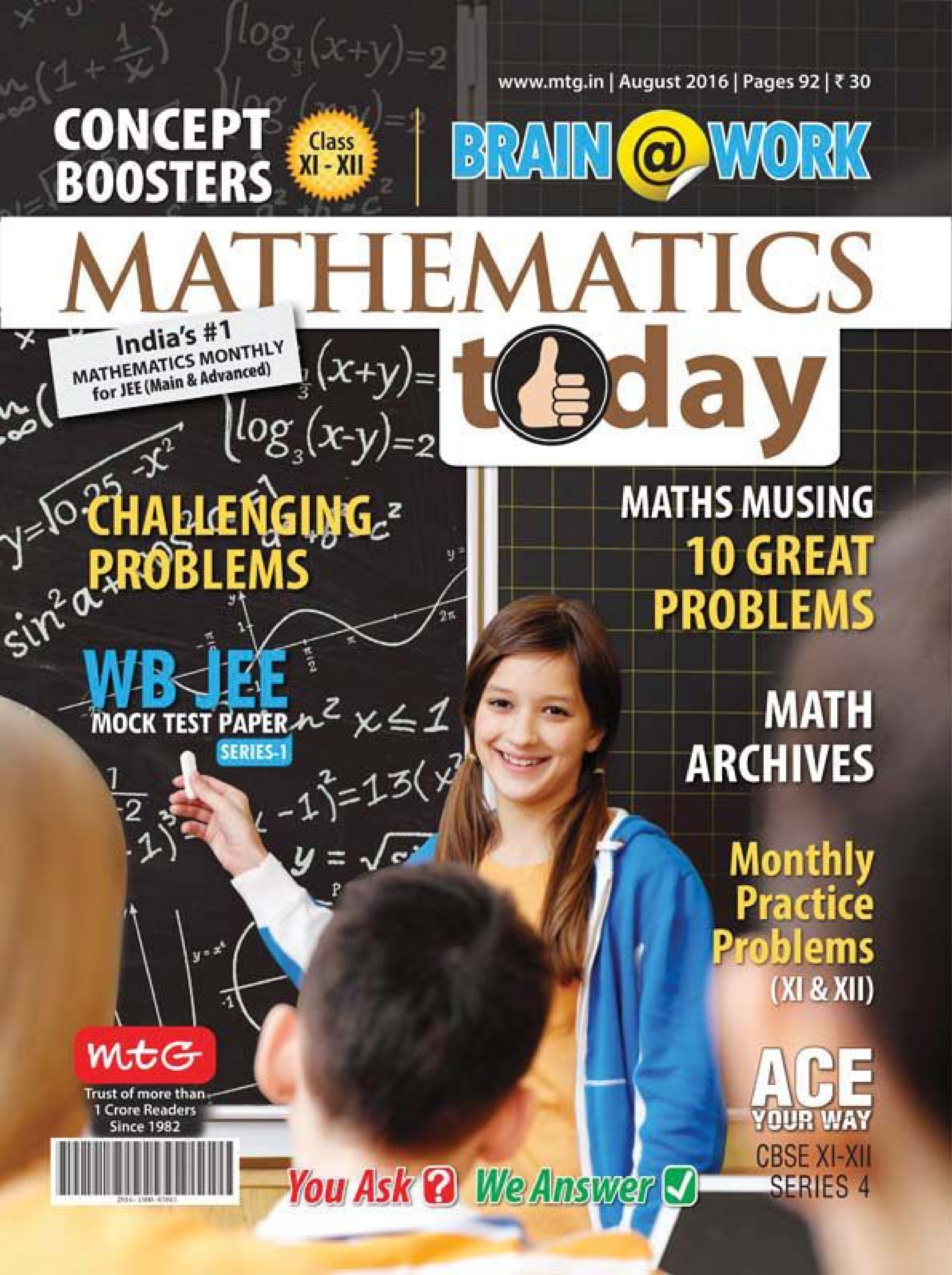
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Vol. XXXIV No. 8 August 2016

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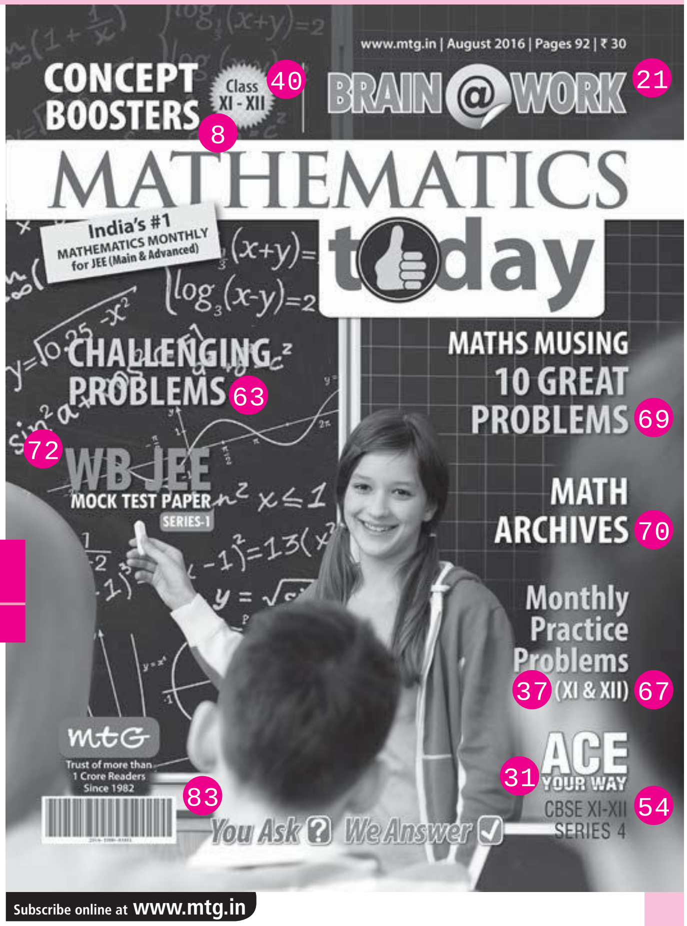

CONTENTS

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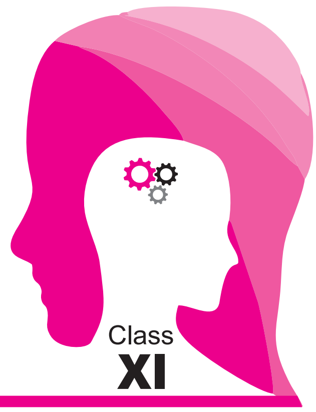


QUADRATIC EQUATIONS

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

The general form of a quadratic equation in x is $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$.

Results

- The solution of the quadratic equation, $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The expression $b^2 - 4ac = D$ is called the discriminant of the quadratic equation.
- If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then
  (i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$
  (iii) $\alpha - \beta = \dfrac{\sqrt{D}}{a}$

NATURE OF ROOTS

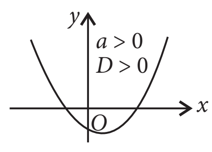
(A) Consider the quadratic equation
$$ax^2 + bx + c = 0,$$
where $a, b, c \in R$ and $a \neq 0$ then
- $D > 0 \Leftrightarrow$ roots are real and distinct (unequal).
- $D = 0 \Leftrightarrow$ roots are real and coincident (equal).
- $D < 0 \Leftrightarrow$ roots are imaginary.
- If $p + iq$ is one root of a quadratic equation, then the other must be the conjugate $p - iq$ and vice versa. ($p, q \in R$ and $i = \sqrt{-1}$)

(B) Consider the quadratic equation
$$ax^2 + bx + c = 0,$$
where $a, b, c \in Q$ and $a \neq 0$ then;
- If $D > 0$ and is a perfect square, then roots are rational and unequal.
- If $\alpha = p + \sqrt{q}$ is one root, in this case, (where p is rational and $\sqrt{q}$ is a surd), then the other root must be the conjugate of it *i.e.*, $\beta = p - \sqrt{q}$ and vice versa.

- A quadratic equation whose roots are α and β, is $(x - \alpha)(x - \beta) = 0$ *i.e.* $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ *i.e.* $x^2 -$ (sum of roots)x + product of roots = 0.
- Remember that a quadratic equation cannot have three different roots and if it has, it becomes an identity.
- Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ and $a, b, c \in R$ then
  - The graph between x, y is always a parabola. If $a > 0$, then the shape of the parabola is concave upwards and if $a < 0$ then the shape of the parabola is concave downwards.
  - $\forall\, x \in R$, $y > 0$ only if $a > 0$ and $b^2 - 4ac < 0$.
  - $\forall\, x \in R$, $y < 0$ only if $a < 0$ and $b^2 - 4ac < 0$.

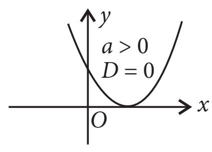
Carefully go through the 6 different shapes of the parabola given as follows.

Fig. 1



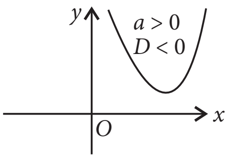
Roots are real and distinct

Fig. 2



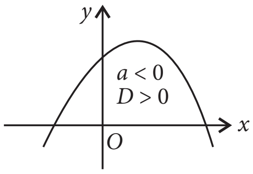
Roots are coincident

Fig. 3



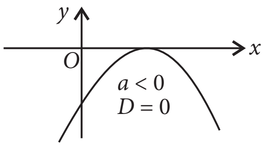
Roots are complex conjugate

Fig. 4



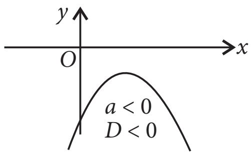
Roots are real and distinct

Fig. 5



Roots are coincident

Fig. 6



Roots are complex conjugate

SOLUTION OF QUADRATIC INEQUALITIES

We have, $ax^2 + bx + c > 0$ $(a \neq 0)$.

- If $D > 0$, then the equation $ax^2 + bx + c = 0$ has two different roots $x_1 < x_2$.
  Then $a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_2, \infty)$
  $a < 0 \Rightarrow x \in (x_1, x_2)$
- If $D = 0$, then roots are equal, *i.e.* $x_1 = x_2$.
  In that case $a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_1, \infty)$
  $a < 0 \Rightarrow x \in \phi$
- Inequalities of the form $\frac{P(x)}{Q(x)} \lesseqgtr 0$ can be quickly solved using the method of intervals.
- Maximum and minimum value of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as; $a < 0$ or $a > 0$.

$$y \in \left[\frac{4ac - b^2}{4a}, \infty\right), \text{ if } a > 0 \text{ and}$$

$$y \in \left[-\infty, \frac{4ac - b^2}{4a}\right), \text{ if } a < 0.$$

COMMON ROOTS OF TWO QUADRATIC EQUATIONS [ONLY ONE COMMON ROOT]

- Let α be the common root of $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$. Therefore
  $a\alpha^2 + b\alpha + c = 0$ and $a'\alpha^2 + b'\alpha + c' = 0$.
  By Cramer's Rule

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

Therefore, $\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

- **The condition that a quadratic function** $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors is that
  $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\text{or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

THEORY OF EQUATIONS

- If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the equation $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ where $a_0, a_1, \ldots, a_n$ are all real and $a_0 \neq 0$, then

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \quad \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0},$$

$$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \quad \alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

REMARKS

- If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.
- Every equation of n^{th} degree $(n \geq 1)$ has exactly n roots and if the equation has more than n roots, it is an identity.
- If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. *i.e.* imaginary roots occur in conjugate pairs.
- If the coefficients in the equation are all rational and $\alpha + \sqrt{\beta}$ is one of its root, then $\alpha - \sqrt{\beta}$ is also a root, where $\alpha, \beta \in Q$ and β is not a perfect square.
- If there be any two real numbers 'a' and 'b' such that $f(a)$ and $f(b)$ are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.
- Every equation $f(x) = 0$ of odd degree has atleast one real root of a sign opposite to that of its last term.

LOCATION OF ROOTS

Let $f(x) = ax^2 + bx + c$, where $a > 0$ and $a, b, c \in R$.

- Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $b^2 - 4ac \geq 0$; $f(d) > 0$ and $(-b/2a) > d$.

- Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'd' (in other words, the number 'd' lies between the roots of $f(x) = 0$) is $f(d) < 0$.
- Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d, e) *i.e.* $d < x < e$ are $b^2 - 4ac > 0$ and $f(d) \cdot f(e) < 0$.
- Conditions that both roots of $f(x) = 0$ to be confined between the numbers p and q $(p < q)$ are $b^2 - 4ac \geq 0$; $f(p) > 0$; $f(q) > 0$ and $p < (-b/2a) < q$.

LOGARITHMIC INEQUALITIES

- For $a > 1$, $0 < x < y \Rightarrow \log_a x < \log_a y$
- For $0 < a < 1$, $0 < x < y \Rightarrow \log_a x > \log_a y$
- If $a > 1$, then $\log_a x < p \Rightarrow 0 < x < a^p$
- If $a > 1$, then $\log_a x > p \Rightarrow x > a^p$
- If $0 < a < 1$, then $\log_a x < p \Rightarrow x > a^p$
- If $0 < a < 1$, then $\log_a x > p \Rightarrow 0 < x < a^p$

PROBLEMS

Single Correct Answer Type

1. If one root is square of the other root of the equation $x^2 + px + q = 0$, $q \in R$, then

(a) $p \leq \frac{1}{4}$ (b) $p \geq \frac{1}{4}$

(c) $-1 \leq p \leq 1$ (d) none of these

2. If α and β are roots of $a(x^2 - 1) + 2bx = 0$ and the quadratic equation whose roots are $2\alpha - \frac{1}{\beta}$ and $2\beta - \frac{1}{\alpha}$ is $px^2 + qx + r = 0$, then $p + q + r$ is equal to

(a) $2b$ (b) $6a - 8b$

(c) $6b - 8a$ (d) 0

3. If p and q are prime numbers and the equation $x^2 - px + q = 0$ has distinct integral roots, then which of the following statements are true?

I. The difference of the roots is odd

II. Atleast one root is prime

III. $p^2 - q$ is prime IV. $p + q$ is prime

(a) I only (b) II only

(c) II and III only (d) all are true

4. Let $p, q \in I$ such that $p^2 + 3p^2q^2 = 30q^2 + 517$, then the value of $3p^2q^2$ is equal to

(a) 169 (b) 488 (c) 588 (d) 688

5. Let a, b, c are all different and non-zero real numbers which are in arithmetic progression. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β such that $\frac{1}{\alpha} + \frac{1}{\beta}$, $\alpha + \beta$ and $\alpha^2 + \beta^2$ are in geometric progression, then the value of $\frac{a}{c}$ is

(a) 1 (b) 2 (c) 3 (d) 4

6. Let $f(x) = x^3 + x + 1$. Suppose $P(x)$ be a cubic polynomial such that $P(0) = -1$ and the zeros of $P(x)$ are the squares of the roots of $f(x) = 0$. Then the value of $P(4)$ is

(a) 100 (b) -99 (c) 99 (d) -100

7. Let $f(x) = x^2 - bx + c$, b is an odd positive integer, $f(x) = 0$ have two prime numbers as roots and $b + c = 35$. Then, the global minimum value of $f(x)$ is

(a) $-\frac{183}{4}$ (b) $\frac{173}{16}$

(c) $-\frac{81}{4}$ (d) data not sufficient

8. If the sum of the roots of the quadratic equation, $ax^2 + bx + c = 0$ is equal to sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

9. A value of b for which the equations $x^2 + bx - 1 = 0$; $x^2 + x + b = 0$, have one root in common, is

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$

10. If $a \in R$ and the equation $(a-2)(x-[x])^2 + 2(x-[x]) + a^2 = 0$ (where $[x]$ denotes G.I.F) has no integral solution and has exactly one solution in $(2, 3)$, then a lies in the interval

(a) $(0, 1)$ (b) $(1, 2)$ (c) $(-1, 3)$ (d) $(-1, 0)$

11. If a, b, c and d are distinct positive real numbers such that a and b are the roots of $x^2 - 10cx - 11d = 0$ and c and d are the roots of $x^2 - 10ax - 11b = 0$, then the value of $a + b + c + d$ is

(a) 1110 (b) 1010 (c) 1101 (d) 1210

12. If the roots of $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then a, b, c are in

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

13. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ $(n \geq 2)$, then $aS_{n+1} + bS_n + cS_{n-1} =$

(a) 0 (b) $a + b + c$

(c) $(a + b + c)^n$ (d) $n^2 abc$

14. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 4 (b) 2 (c) 0 (d) –2

15. The value of a for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as other, is

(a) –2/3 (b) 1/3 (c) –1/3 (d) 2/3

16. If $x^2 + 2ax + 10 - 3a > 0$ for each $x \in R$, then

(a) $a < -5$ (b) $-5 < a < 2$
(c) $a > 5$ (d) $2 < a < 5$

17. If a, b, c are positive numbers such that $a > b > c$ and the equation

$(a + b - 2c)x^2 + (b + c - 2a)\ x + (c + a - 2b) = 0$

has a root in the interval (–1, 0), then

(a) b cannot be the G.M. of a, c
(b) b may be the G.M. of a, c
(c) b is the G.M. of a, c
(d) none of these

Multiple Correct Answer Type

18. If a and b are two numbers such that $a^2 + b^2 = 7$ and $a^3 + b^3 = 10$, then

(a) greatest value of $|a + b|$ is 5
(b) greatest value of $a + b$ is 4
(c) least value of $a + b$ is 1
(d) least value of $|a + b|$ is 1

19. Let $f(x) = |x - 1| + |x - 2| + |x - 3| + |x - 4|$, then

(a) the least value of $f(x)$ is 4
(b) the least value is not attained at a unique point
(c) the number of integral solution of $f(x) = 4$ is 2
(d) the value of $\dfrac{f(\pi-1)+f(e)}{2f(12/5)}$ is 1

20. The roots of the equation $x^2 + 2(a - 3)x + 9 = 0$ lie between –6 and 1 and 2, $h_1, h_2, ..., h_{20}$, $[a]$ are in H.P., where $[a]$ denotes the integral part of a and 2, $a_1, a_2, ..., a_{20}$, $[a]$ are in A.P., then

(a) $h_{18} = \dfrac{14}{3}$ (b) $a_3 = \dfrac{18}{7}$
(c) $[a] = 6$ (d) $a_3 h_{18} = 11$

21. The value(s) of a for which

$$\frac{x^3 - 6x^2 + 11x - 6}{x^3 + x^2 - 10x + 8} + \frac{a}{30} = 0$$

does not have a real solution is/are

(a) –10 (b) 12 (c) 5 (d) –30

22. If the equation $(a + 2)x^2 + bx + c = 0$ and $2x^2 + 3x + 4 = 0$ have a common root where $a, b, c \in N$ then

(a) $b^2 - 4ac < 0$
(b) minimum value of $a + b + c$ is 16
(c) $b^2 < 4ac + 8c$
(d) minimum value of $a + b + c$ is 7

23. If the function f satisfies $f\left(\dfrac{2x-1}{x+1}\right) = 2x$, then

(a) $f(1) = 4$
(b) $f(x) + f(-x) = 0$ has no real roots
(c) $f(x) + f(-x)$ has only real roots
(d) sum of the roots of $f(x) + f(-x) = 0$ is zero

24. For real x, the function $\dfrac{(x-a)(x-b)}{x-c}$ will assume all real values provided

(a) $a > b > c$ (b) $a < b < c$
(c) $a > c > b$ (d) $a < c < b$

25. If the quadratic equation $ax^2 + bx + c = 0$ $(a > 0)$ has $\sec^2\theta$ and $\operatorname{cosec}^2\theta$ as its roots, then which of the following hold good?

(a) $b + c = 0$ (b) $b^2 - 4ac \geq 0$
(c) $c \geq 4a$ (d) $4a + b \geq 0$

26. If $\cos\alpha$ is a root of the equation $169x^2 - 26x - 35 = 0$, $-1 < x < 0$, then $\sin 2\alpha$ is

(a) $\dfrac{144}{169}$ (b) $-\dfrac{144}{169}$ (c) $\dfrac{120}{169}$ (d) $-\dfrac{120}{169}$

27. The equation $(ay - bx)^2 + 4xy = 0$ has rational solutions x, y for

(a) $a = 1/2, b = 2$ (b) $a = 4, b = 1/8$
(c) $a = 1, b = 3/4$ (d) $a = 2, b = 1$

28. The real values of 'a' for which the equation $x^4 - 2x^2a - x^2 + 6x + a^2 - 3a = 0$ has all its roots real

(a) $a > 4$ (b) $a \geq 2$ (c) $a \geq 3/4$ (d) $a \geq -1$

29. Let 'm' be a real number, and suppose that two of the three solutions of the cubic equation $x^3 + 3x^2 - 34x = m$ differ by 1. Then possible values of 'm' is /are

(a) 120 (b) 80 (c) – 48 (d) –32

30. The inequality $\left|\dfrac{x^2+kx+1}{x^2+x+1}\right| < 3$ is satisfied for all real values of x, then

(a) $k \in (-1, 5)$
(b) the number of integral values of k is 5
(c) $k \in (-5, 1)$
(d) the equation $x^2 - (k^2 - 4k - 5)x - 1 = 0$ has one real root in (0, 1)

Comprehension Type

Paragraph for Question 31 to 33

Let $x, y, z, w \in R$ satisfy the system of equations

$x + y + z + w = 1$

$x + 2y + 4z + 8w = 16$

$x + 3y + 9z + 27w = 81$

$x + 4y + 16z + 64w = 256$, then

31. The number which is not divisible by 5 is

(a) w (b) $z + x$

(c) y (d) $w + y + z$

32. The number of common divisors to y and w is/are

(a) 4 (b) 3 (c) 1 (d) 0

33. The pair which is co-prime is

(a) $(|w|, |z|)$ (b) $(|z|, |y|)$

(c) $(|y|, |x|)$ (d) $(|z|, |x|)$

Paragraph for Question No. 34 to 36

Let $f(x) = ax^2 + bx + c$, where a, b and c are real and $a \neq 0$. Let $\alpha < \beta$ be the roots of $f(x) = 0$. Then

(A) for all x such that $\alpha < x < \beta$, $f(x)$ and a have opposite signs.

(B) for $x < \alpha$ or $x > \beta$, $f(x)$ and a have the same sign.

34. If both the roots of the equation $x^2 - mx + 1 = 0$ are less than unity, then

(a) $m > 2$ (b) $-1 \leq m \leq 3$

(c) $0 \leq m \leq 5/2$ (d) none of these

35. If both the roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ are greater than 3, then

(a) $m < 0$ (b) $m > 1$

(c) $0 < m < 1$ (d) $m > 11/9$

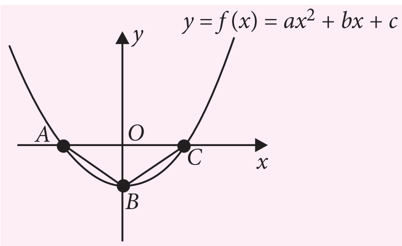
36. If both the roots of the equation $4x^2 - 2x + m = 0$ belong to the interval $(-1, 1)$, then

(a) $-2 \leq m \leq \frac{3}{4}$ (b) $0 < m < 2$

(c) $2 < m < \frac{5}{2}$ (d) $-2 < m < \frac{1}{4}$

Paragraph for Question No. 37 to 39

In the given figure, vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. The ΔABC is right angled isosceles triangle, whose hypotenuse $AC = 4\sqrt{2}$ units.



37. $y = f(x)$ is given by

(a) $y = x^2 - 2\sqrt{2}$ (b) $y = x^2 - 12$

(c) $y = \frac{x^2}{2} - 2$ (d) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

38. The minimum value of $y = f(x)$ is

(a) -4 (b) -2 (c) $-2\sqrt{2}$ (d) -12

39. Number of integral values of 'K' for which one root of $f(x) = 0$ is more than 'K' and other less than 'K' is

(a) 4 (b) 5 (c) 6 (d) 7

Matrix-Match Type

40. Match the following.

	Column-I		Column-II
(A)	If $x^2 + x - a = 0$ has integral roots and $a \in N$, then a can be equal to	(p)	2
(B)	If the equation $ax^2 + 2bx + 4c = 16$ has no real roots and $a + c > b + 4$ then integral value of c may be equal to	(q)	12
(C)	If $x^2 + 2bx + 9b - 14 = 0$, has only negative roots, then integral values of b may be	(r)	3
(D)	If A is number of solutions of $\lvert x^2 - x - 6\rvert = x + 2$ then A is	(s)	20

Integer Answer Type

41. If $[x]$ denotes the greatest integer less than or equal to x, then the number of integer values of x which satisfies the equation

$[x^2 - 2]^2 - 9[x^2 - 2] + 14 = 0$ is

42. If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + 2 = 0$ have a common root and $a, b \in R$, and the value of $a + b$ is equal to $k/5$, then $k =$

43. If the equation $x^2 + 2\alpha x + \alpha^2 - 1 = 0$ and $x^2 + 2\beta x + \beta^2 - 1 = 0$ have a common root $(\alpha \neq \beta)$, then the value of the expression $2\alpha^2 - 4\alpha\beta - |\alpha - \beta| + |\alpha - \beta|\beta^2$, is

44. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

45. If $f(x)$ is a polynomial of degree four with leading coefficient one satisfying $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, then $\left[\frac{f(-1)+f(5)}{f(0)+f(4)}\right]$ (where $[\cdot]$ represents greatest integer function) is equal to

46. Let t be any root of $x^n + x^{n-1} + x^{n-2} + ... + x + 1 = 0$ then $\frac{(t^{2n+2}+3)(4-t^{3n+3})}{4} =$

47. The number of integral solutions of equation $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)=4$ is (where $a, b \in I$).

48. The number of the distinct real roots of the equation $(x + 1)^5 = 2(x^5 + 1)$ is

49. The set of real parameter 'a' for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all real solutions, is given by $\left[\frac{m}{n}, \infty\right)$ where m and n are relatively prime positive integers, then the value of $(m + n)$ is

50. If α, β be the roots of $x^2 + px - q = 0$ and γ, δ are the roots of $x^2 + px + r = 0$, $q + r \neq 0$, then $\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}$ is equal to

SOLUTIONS

1. (a) : $p^3 - (3p - 1)q + q^2 = 0$, $q \in R$

$D \geq 0$

$(3p - 1)^2 - 4p^3 \geq 0 \Rightarrow 4p^3 - (9p^2 + 1 - 6p) \leq 0$

$\Rightarrow (p-1)^2(4p-1) \leq 0 \Rightarrow p \leq \frac{1}{4}$

2. (c) : We get the quadratic equation as

$ax^2 + 6bx - 9a = 0$

So $p + q + r = 6b - 8a$

3. (d) : Product of the roots $= q \Rightarrow$ roots are 1 and q

$\Rightarrow p = q + 1$ is prime $\Rightarrow q = 2$ and $p = 3$.

4. (c) : $(p^2 - 10)(3q^2 + 1) = 3 \times 13^2$

$\Rightarrow (p^2 - 10) = 39$ and $(3q^2 + 1) = 13$

5. (c) : $a + c = 2b$ and $2ac - b^2 = bc$. Eliminating 'b' from these two equations, we get

$a^2 - 4ac + 3c^2 = 0 \Rightarrow a = 3c \ (\because a \neq c)$

6. (c) : Let $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$

α, β, γ are roots of $f(x) = 0$

$P(x) = k(x-\alpha^2)(x-\beta^2)(x-\gamma^2)$

$x = 0 \Rightarrow P(0) = -1 = -k\alpha^2\beta^2\gamma^2 \Rightarrow k\alpha^2\beta^2\gamma^2 = 1$

We have $\alpha\beta\gamma = -1 \therefore k = 1$

Now, $P(x^2) = (x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)$

$= (x-\alpha)(x-\beta)(x-\gamma)(x+\alpha)(x+\beta)(x+\gamma) = -f(x)f(-x)$

$\therefore P(4) = -f(2)f(-2) = -(11)(-9) = 99$

7. (c) : Let α, β be the roots of $x^2 - bx + c = 0$,

Then, $\alpha + \beta = b$

$\Rightarrow$ one of the roots is '2' (Since α, β are primes and b is odd positive integer)

$\Rightarrow f(2) = 0 \Rightarrow 2b - c = 4$ and $b + c = 35$

$\Rightarrow b = 13, c = 22$

$\therefore$ Minimum value $= f\left(\frac{13}{2}\right) = -\frac{81}{4}$

8. (c) : Let the roots of the equation $ax^2 + bx + c = 0$ are α and β

$\alpha+\beta = \frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}$

$\Rightarrow (\alpha + \beta)(\alpha\beta)^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\Rightarrow \left(-\frac{b}{a}\right)\left(\frac{c^2}{a^2}\right) = \frac{b^2}{a^2} - \frac{2c}{a} \Rightarrow -bc^2 = ab^2 - 2a^2c$

$\Rightarrow ab^2 + bc^2 = 2a^2c$

Dividing by abc, we get

$\frac{b}{c}+\frac{c}{a}=\frac{2a}{b} \Rightarrow \frac{b}{c}, \frac{a}{b}, \frac{c}{a}$ are in A.P.

$\Rightarrow \frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

9. (b) : Using condition for common root $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$, we have

$(-1 - b)^2 = (1 - b)(b^2 + 1)$

$\Rightarrow b^3 + 3b = 0 \Rightarrow b = 0, \pm i\sqrt{3}$

10. (d) : $(a - 2)(x - [x])^2 + 2(x - [x]) + a^2 = 0$ ... (1)

$f(y) = (a - 2)y^2 + 2y + a^2 = 0$

where $y = x - [x]$ ...(2)

Since x cannot be an integer

When $2 < x < 3 \Rightarrow [x] = 2$ and $0 < y < 1$

$\therefore$ Equation (1) has exactly one solution in (2, 3)

$\therefore$ Equation (2) has exactly one solution in (0, 1)

$\therefore f(0) \cdot f(1) < 0 \Rightarrow a^2(a - 2 + 2 + a^2) < 0$

$\Rightarrow a(a + 1) < 0 \Rightarrow a \in (-1, 0)$

11. (d) : Since a and b are the roots of $x^2 - 10cx - 11d = 0$, we have

$a + b = 10c$ ...(i) and $ab = -11d$ ...(ii)

Also, since c and d are the roots of $x^2 - 10ax - 11b = 0$, we have

$c + d = 10a$ ...(iii) and $cd = -11b$ ...(iv)

Adding (i) and (iii), we get

$a + b + c + d = 10(a + c)$

$\Rightarrow b + d = 9(a + c)$ ...(1)

Multiplying (ii) and (iv), we get

$abcd = 121bd \Rightarrow ac = 121$ ...(2)

Also, $a^2 - 10ca - 11d = 0 = c^2 - 10ca - 11b$

$\Rightarrow a^2 + c^2 - 20ca - 11(b + d) = 0$

From (1) and (2), we have

$a^2 + c^2 - 20(121) - 99(a + c) = 0$

$\Rightarrow (a + c)^2 - 2 \times 121 - 20 \times 121 - 99(a + c) = 0$

$\Rightarrow (a + c - 121)(a + c + 22) = 0$

$\Rightarrow a + c = 121; \ a + c = -22$

Since a, c are positive, $a + c \neq -22$.

$\Rightarrow a + c = 121$ and

$a + b + c + d = 1210$

12. (c) : $x = 1$ is a root $\Rightarrow$ other root is also 1

$\therefore$ product of roots $=1$

$$\Rightarrow \frac{c(a-b)}{a(b-c)} = 1 \Rightarrow 2ac = ab + bc \Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

13. (a) : $S_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$$= -\frac{b}{a} \cdot S_n - \frac{c}{a} \cdot S_{n-1}$$

14. (c) : Since α, β, γ are the roots of $x^3 + px + q = 0$

$\therefore \ \alpha + \beta + \gamma = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

15. (d) : $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ ...(i)

Let α and 2α be the roots of (i), then

$(a^2 - 5a + 3)\alpha^2 + (3a - 1)\alpha + 2 = 0$ ...(ii)

and $(a^2 - 5a + 3)(4\alpha^2) + (3a - 1)(2\alpha) + 2 = 0$ ...(iii)

Multiplying (ii) by 4 and subtracting it from (iii), we get $(3a - 1)(2\alpha) + 6 = 0$

Clearly $a \neq 1/3$. Therefore, $\alpha = -3/(3a - 1)$

Putting this value in (ii), we get

$(a^2 - 5a + 3)(9) - (3a - 1)^2(3) + 2(3a - 1)^2 = 0$

$\Rightarrow 9a^2 - 45a + 27 - (9a^2 - 6a + 1) = 0$

$\Rightarrow -39a + 26 = 0 \ \Rightarrow a = 2/3$

For $a = 2/3$, the equation becomes $x^2 + 9x + 18 = 0$ whose roots are $-3, -6$.

16. (b) : $x^2 + 2ax + 10 - 3a > 0 \ \forall \ x \in R$

$\Rightarrow D = 4a^2 - 4(10 - 3a) < 0$

$\Rightarrow a^2 + 3a - 10 < 0 \ \Rightarrow (a + 5)(a - 2) < 0$

$\Rightarrow -5 < a < 2$

17. (a) : Let $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$

According to the given condition, we have

$f(0)\,f(-1) < 0$

i.e. $(c + a - 2b)(2a - b - c) < 0$

i.e. $(c + a - 2b)(a - b + a - c) < 0$

i.e. $c + a - 2b < 0$

$[a > b > c$, given $\Rightarrow a - b > 0, a - c > 0]$

i.e. $b > \dfrac{a+c}{2}$

$\Rightarrow$ b cannot be the G.M. of a and c, since G.M. < A.M. always.

18. (a,b,d) : Let $a + b = x$, then $ab = \dfrac{x^2 - 7}{2}$

and $a^3 + b^3 = 10 \Rightarrow (a+b)^3 - 3ab(a+b) = 10$

$\Rightarrow x^3 - 21x + 20 = 0 \ \Rightarrow x = 1, 4, -5$

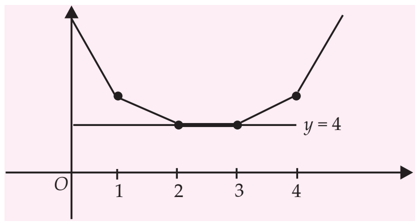
19. (a,b,c,d) : $f(x) = 10 - 4x$; if $-\infty < x < 1$

$= 8 - 2x$; if $1 < x \leq 2$

$= 4$; if $2 < x \leq 3$

$= 2x - 2$; if $3 < x \leq 4$

$= 4x - 10$; if $4 < x < \infty$



Clearly the least value of $f(x)$ is 4

The number of integral solutions of $f(x) = 4$ are two.

Since $\pi - 1, e, \dfrac{12}{5} \in [2,3] \ \therefore \ f(\pi - 1) = f(e) = f\left(\dfrac{12}{5}\right) = 4$

$$\therefore \ \frac{f(\pi - 1) + f(e)}{2f(12/5)} = 1$$

20. (a,b,c) : According to the given conditions, we have

(1) $D \geq 0$

(2) $-6 < \dfrac{-b}{2a} < 1$

(3) $f(-6)f(1) > 0$

$\therefore \ 6 \leq a < 6.75$

$[a] = 6$

i.e., $2, h_1, h_2, ..., h_{20}, [a]$ are in H.P.

$2, h_1, h_2, ..., h_{20}, 6$ are in H.P.

$$\therefore \ \frac{1}{h_{18}} = \frac{1}{2} + 18\left[\frac{\frac{1}{6} - \frac{1}{2}}{21}\right] \Rightarrow h_{18} = \frac{14}{3}$$

$2, a_1, a_2, ... a_{20}, 6$ are in A.P.

$$\therefore \ a_3 = 2 + 3\left(\frac{6-2}{21}\right) = \frac{18}{7}$$

21. (b,c,d): $\dfrac{x^3-6x^2+11x-6}{x^3+x^2-10x+8}=\dfrac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+4)}$

$\therefore\ x \neq 1, 2, -4$ then $f(x)=\dfrac{x-3}{x+4}$

Range of $f(x)=R-\left\{1,-\dfrac{2}{5},-\dfrac{1}{6}\right\}$

So equation does not have a solution if $\dfrac{a}{30}=-1,\dfrac{2}{5},\dfrac{1}{6}$

$\Rightarrow\ a = -30, 12, 5$

22. (b, c) : $(a + 2)x^2 + bx + c = 0$ ...(i)

$2x^2 + 3x + 4 = 0$ ...(ii)

For (ii) $D < 0 \Rightarrow$ (i) and (ii) have both roots common.

$\therefore\ \dfrac{a+2}{2}=\dfrac{b}{3}=\dfrac{c}{4}$

$\therefore$ For equation (i) $D < 0$

$\Rightarrow\ b^2-4c(a+2)<0\ \Rightarrow\ b^2<4ac+8c$

$\dfrac{a+2}{2}=\dfrac{b}{3}=\dfrac{c}{4}=k$ (let) $\Rightarrow\ k\in N,\ k\geq 2$

$\dfrac{a+b+c+2}{9}=k\ \Rightarrow\ a+b+c=9k-2$

$(a + b + c)_{min} = 16$

23. (a,b,d): $f(t)=2\times\left(\dfrac{t+1}{2-t}\right)$

$\Rightarrow\ f(x)=2\left(\dfrac{x+1}{2-x}\right)\Rightarrow f(-x)=2\left(\dfrac{-x+1}{2+x}\right)=2\left(\dfrac{1-x}{x+2}\right)$

Let $t=\dfrac{2x-1}{x+1}\ \Rightarrow\ tx+t=2x-1$

$\Rightarrow\ (t-2)x=-(1+t)\ \Rightarrow\ x=\dfrac{t+1}{2-t}$

Now, $f(x)+f(-x)=0\ \Rightarrow\ \dfrac{x+1}{2-x}+\dfrac{1-x}{x+2}=0$

$\Rightarrow\ (x^2+3x+2)+(x^2-3x+2)=0$

$\Rightarrow\ x^2+2=0\ \Rightarrow\ x=\sqrt{2}i,-\sqrt{2}i$

24. (c,d) : Let $y=\dfrac{(x-a)(x-b)}{(x-c)}$

$\Rightarrow\ (x - c)y = x^2 - (a + b)x + ab$

$\Rightarrow\ x^2 - (a + b + y)x + ab + cy = 0$

Here, $D = (a + b + y)^2 - 4(ab + cy)$

$= y^2 + 2y(a + b - 2c) + (a - b)^2$

Since x is real and y assumes all real values,

$\therefore\ D \geq 0$ for all real values of y

$\Rightarrow\ y^2 + 2y(a + b - 2c) + (a - b)^2 \geq 0$

Now, we know that the sign of a quadratic is same as that of coeff. of y^2 provided its discriminant

$B^2 - 4AC < 0$

This will be so if,

$4(a + b - 2c)^2 - 4(a - b)^2 < 0$

or $4(a + b - 2c + a - b)(a + b - 2c - a + b) < 0$

$\Rightarrow\ 16(a - c)(b - c) < 0 \Rightarrow 16(c - a)(c - b) < 0$ ...(1)

Now, If $a < b$ then from inequation (1), we get

$c \in (a, b)\ \Rightarrow\ a < c < b$

or If $a > b$ then from inequation (1), we get $c \in (b, a)$

$\Rightarrow\ b < c < a$ or $a > c > b$

25. (a,b,c) : Sum of roots = product of roots

$\Rightarrow\ -\dfrac{b}{a}=\dfrac{c}{a}\ \Rightarrow\ b+c=0$

$\therefore\ \sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \cdot \operatorname{cosec}^2\theta$

Roots are real, $D \geq 0$

$b^2 - 4ac \geq 0$ but $b = -c$, product of the roots > 0

$\Rightarrow\ c^2 - 4ac \geq 0\ \Rightarrow\ c(c - 4a) \geq 0\ \because\ c > 0$

$\Rightarrow\ c - 4a \geq 0$

$\Rightarrow\ c \geq 4a\ \Rightarrow\ -b \geq 4a\ \Rightarrow\ 4a + b \leq 0$

26. (c,d): $(13x - 1)^2 = 36\ \Rightarrow\ 13x - 1 = \pm 6$

$\Rightarrow\ x=\dfrac{7}{13}$ or $x=-\dfrac{5}{13}$

But $x\neq\dfrac{7}{13}$ as $-1<x<0\ \therefore\ x=-\dfrac{5}{13}$

$\Rightarrow\ \cos\alpha=-\dfrac{5}{13};\ \sin\alpha=\pm\dfrac{12}{13}$

$\Rightarrow\ \sin 2\alpha=2\sin\alpha\cos\alpha=\pm\dfrac{120}{169}$

27. (a,c) : Given equation is $(ay - bx)^2 + 4xy = 0$

$\Rightarrow\ a^2y^2 + b^2x^2 + (4 - 2ab)xy = 0$

$\Rightarrow\ \dfrac{a^2y}{x}+\dfrac{b^2x}{y}-(2ab-4)=0$ ...(i)

Put $\dfrac{y}{x}=t$ in (i), we get

$\Rightarrow\ a^2t^2 - (2ab - 4)t + b^2 = 0$

$\Rightarrow\ D = (2ab - 4)^2 - 4a^2b^2 = 16(1 - ab)$

must be a perfect square for rational roots.

28. (b,d): Assuming quadratic in a

$a = x^2 + 2x,\ a = x^2 - 2x + 3$

$\therefore$ Given exp. is $(a - x^2 - 2x)(a - x^2 + 2x - 3) = 0$

$\therefore\ x=-1\pm\sqrt{1+a}\ \ \&\ \ x=1\pm\sqrt{a-2}$

$\Rightarrow\ a \geq -1,\ a \geq 2$

29. (a,c) : Suppose that both r and $r + 1$ are solutions of the equation $x^3 + 3x^2 - 34x = m$

Then $r^3 + 3r^2 - 34r = m$ and

$(r + 1)^3 + 3(r + 1)^2 - 34(r + 1) = m$

Subtracting the first of these equations from the second yields $(3r^2 + 3r + 1) + 3(2r + 1) - 34 = 0$, and simplification yields $3r^2 + 9r - 30 = 0$.

Thus $0 = r^2 + 3r - 10 = (r + 5)(r - 2)$

We conclude that either $r = -5$ or $r = 2$.

If $r = -5$, then $m = (-5)^3 + 3(-5)^2 - 34(-5)$
$= -125 + 75 + 170 = 120$

The other possibility is $r = 2$, which yields $m = 8 + 12 - 68 = -48$. $\therefore$ $m = 120$ and $m = -48$ are the two possibilities.

30. (a,b,d) : $-3<\dfrac{x^2+kx+1}{x^2+x+1}<3$

$\Rightarrow$ $-3(x^2 + x + 1) < x^2 + kx + 1 < 3(x^2 + x + 1)$

$\Rightarrow$ $4x^2 + (k + 3)x + 4 > 0$ and
$2x^2 + (3 - k)x + 2 > 0$ $\forall\, x \in R$

$\Rightarrow$ $(k + 3)^2 - 64 < 0$ and $(k - 3)^2 - 16 < 0$

$\Rightarrow$ $k^2 + 6k - 55 < 0$ and $k^2 - 6k - 7 < 0$

$\Rightarrow$ $k \in (-11, 5)$ and $k \in (-1, 7)$ $\Rightarrow$ $k \in (-1, 5)$

The number of integral values of k is 5

If $f(x) = x^2 - (k^2 - 4k - 5)x - 1$

Then $f(0) \times f(1) = -1(1 - (k^2 - 4k - 5) - 1)$

$= k^2 - 4k - 5 = (k + 1)(k - 5) < 0$

$\therefore$ $f(x) = 0$ has exactly one real root in $(0, 1)$

(31 - 33) :

31. (b) **32. (a)** **33. (d)**

On solving the given system of equations, we get
$x = -24;$ $y = 50;$ $z = -35;$ $w = 10$.

34. (d) : $f(x) = x^2 - mx + 1$ and $\alpha < \beta$ are the roots of $f(x) = 0$. Now, $\alpha < \beta < 1$ implies that $f(1)$ and the coefficients of x^2 have the same sign. This gives

$1 - m + 1 > 0 \Rightarrow m < 2$ ...(1)

Also, discriminant is $m^2 - 4 \geq 0$.

$\Rightarrow$ $m \leq -2$ or $m \geq 2$ ...(2)

From Eqs. (1) and (2), $m \leq -2$.

Also, note that if $m = -2$, the roots are $-1, -1$.

35. (d) : Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$.

Let $\beta > \alpha > 3$ be the roots of $f(x) = 0$.

Then $6 < \alpha + \beta = 6m$ and hence $m > 1$.

Also, $9m^2 - 2m + 2 = \alpha\beta > 9$

Therefore, $9m^2 - 2m - 7 > 0$

$\Rightarrow$ $(9m + 7)(m - 1) > 0$

This gives, $m<\dfrac{-7}{9}$ or $m>1$

Also, $f(3)$ and the coefficient of x^2 have the same sign. Therefore, $f(3) > 0$. This gives

$9 - 18m + 9m^2 - 2m + 2 > 0$

$\Rightarrow$ $9m^2 - 20m + 11 > 0 \Rightarrow (9m - 11)(m - 1) > 0$

$\Rightarrow$ $m<1$ or $m>\dfrac{11}{9}$, we get $\dfrac{11}{9}<m$

36. (d) : Let α, β, where $\alpha < \beta$, be the roots of $4x^2 - 2x + m = 0$.

Then $-1 < \alpha < \beta < 1$ and $\alpha+\beta=\dfrac{1}{2}, \alpha\beta=\dfrac{m}{4}$

Now, $f(-1)$ and the coefficient of x^2 have the same sign. Therefore, $f(-1) > 0$ and hence

$4 + 2 + m > 0 \Rightarrow m > -6$ ...(1)

Also, $f(1) > 0$ $\Rightarrow$ $4 - 2 + m > 0 \Rightarrow m > -2$ ...(2)

The discriminant is $4 - 16m > 0$.

$\Rightarrow$ $m < 1/4$ ...(3)

From Eqs. (1) to (3), we get $-2<m<\dfrac{1}{4}$

Therefore, the roots are $1/4, 1/4 \in (-1, 1)$.

If the roots are distinct, then $< -2 < m < 1/4$.

37. (d) : Co-ordinate of A is $(-2\sqrt{2}, 0)$

Co-ordinate of C is $(2\sqrt{2}, 0)$

$AB = BC = 4$ units, $OB=2\sqrt{2}$ units

$\therefore$ $y=\dfrac{x^2}{2\sqrt{2}}-2\sqrt{2}$

38. (c) : Minimum value of $y = f(x)$ is $-2\sqrt{2}$ at $x = 0$

39. (b) : $f(x) = 0 \Rightarrow \dfrac{x^2}{2\sqrt{2}}-2\sqrt{2}=0$

$\Rightarrow$ $x^2-8=0$ $\Rightarrow$ $x=\pm 2\sqrt{2}$

Number of integral values of 'K' which lies in $(-2\sqrt{2}, 2\sqrt{2})$ is 5.

40. A $\to$ p, q, s ; B $\to$ q, s ; C $\to$ p, q, s ; D $\to$ r

(A) Discriminant $= 1 + 4a$

$1 + 4a$ must be perfect square

$1+4a=(2\lambda+1)^2; \lambda \in N$ (as $1 + 4a$ is odd)

$a=\lambda(\lambda+1)$

(B) $f(x)=ax^2+2bx+4c-16=0$

$f(-2)=4a-4b+4c-16$ $\Rightarrow$ $4(a+c-b-4)$

$f(-2)>0$ as $(a+c>b+4)$

$f(0)>0$ $\Rightarrow$ $4c-16>0$ $\Rightarrow$ $c>4$

(C) As both roots are negative so

$2b > 0$ and $9b - 14 > 0$ ...(1)

$D \geq 0 \Rightarrow 4b^2 - 4(9b-14) \geq 0$ ...(2)

$\Rightarrow b^2 - 9b + 14 \geq 0 \Rightarrow (b-2)(b-7) \geq 0$

$\Rightarrow b \in (-\infty, 2] \cup [7, \infty)$

From (1) and (2) $b \in [7, \infty) \cup \left(\frac{14}{9}, 2\right]$

(D) $|x^2 - x - 6| = x + 2 \Rightarrow |(x-3)(x+2)| = x + 2$

$$\Rightarrow \begin{cases} x = 4 & ; \quad x < -2 \\ x = -2, 2 & ; \quad x \in [2, 3) \\ x = 4 & ; \quad x \geq 3 \end{cases}$$

$\therefore \; N = 3$

41. (4) : $t^2 - 9t + 14 = 0$

$\Rightarrow (t - 2)(t - 7) = 0$

$\Rightarrow [x^2 - 2] = 2$ or $[x^2 - 2] = 7$

$\Rightarrow 2 \leq x^2 - 2 < 3$ or $7 \leq x^2 - 2 < 8$

$\Rightarrow 4 \leq x^2 < 5$ or $9 \leq x^2 < 10$

$\Rightarrow 2 \leq |x| < \sqrt{5}$ or $3 \leq |x| < \sqrt{10}$

$\Rightarrow x \in (-\sqrt{5}, -2] \cup [2, \sqrt{5})$

or $x \in (-\sqrt{10}, -3] \cup [3, \sqrt{10})$

$\Rightarrow$ The integral values of x are $-2, 2, -3, 3$

42. (8) : Discriminant of $x^2 + 3x + 5 = 0$ is negative. Therefore the roots of $x^2 + 3x + 5 = 0$ are complex conjugate pairs. Since the equations have a common root, the common root is complex and hence both the (complex) roots are common. Therefore the coefficients of the equations are proportional.

$\therefore \; \frac{a}{1} = \frac{b}{3} = \frac{2}{5} \Rightarrow a = \frac{2}{5}$ and $b = \frac{6}{5}$

$\therefore \; a + b = \frac{8}{5}$

43. (6) : Subtracting the two equations, we get the common root as $x = -\frac{1}{2}(\alpha + \beta)$. Substituting this in any equation, we get $(\alpha - \beta)^2 = 4$ *i.e.* $|\alpha - \beta| = 2$

Now, $2\alpha^2 - 4\alpha\beta - |\alpha - \beta| + |\alpha - \beta|\beta^2$

$= 2\alpha^2 - 4\alpha\beta + 2\beta^2 - 2 = 2(\alpha - \beta)^2 - 2$

$= 2 \cdot 4 - 2 = 8 - 2 = 6$

44. (4) : $|x|^2 - 3|x| + 2 = |x - 1|\,|x - 2| = 0$

$\Rightarrow |x| = 1, 2$

So, there are four solutions.

45. (5) : According to question,

$$f(x) - x = (x-1)(x-2)(x-3)(x-\alpha)$$

$\Rightarrow f(-1) = 24(1+\alpha) - 1$

$f(0) = 6\alpha$

$f(4) = 6(4 - \alpha) + 4$

$f(5) = 24(5 - \alpha) + 5$

$\Rightarrow \left[\frac{f(-1) + f(5)}{f(0) + f(4)}\right] = \left[\frac{148}{28}\right] = 5$

46. (3) : $x = n^{\text{th}}$ root of unity $\Rightarrow t_n = 1$

47. (1) : $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = 4$, for $a, b \in I$

$a \in I \Rightarrow \frac{1}{a} \leq 1$

$\Rightarrow 1 + \frac{1}{a} \leq 2 \Rightarrow \left|1 + \frac{1}{a}\right| \leq 2$ and $\left|1 + \frac{1}{b}\right| \leq 2$

$\Rightarrow \left|1 + \frac{1}{a}\right| . \left|1 + \frac{1}{b}\right| \leq 4$

This is true only when $a = 1$ and $b = 1$

$\therefore$ Number of integral solutions is one.

48. (3) : $(x + 1)^5 = 2(x^5 + 1)$

Let $f(x) = \frac{(x+1)^5}{(x^5+1)} \quad (x \neq -1)$

$\Rightarrow f'(x) = \frac{5(x+1)^4(1-x^4)}{(x^5+1)^2}$

$\Rightarrow x = 1$ is maximum.

As, $f(0) = 1$ and $f(1) = 16$

And $\lim_{x \to \pm\infty} f(x) = 1 \Rightarrow f(x) = 2$ has two solutions but given equation has three solutions because $x = -1$ included.

49. (7)

50. (1) : Here, $\alpha + \beta = -p = \gamma + \delta$

$(\alpha - \gamma)(\alpha - \delta) = \alpha^2 - \alpha(\gamma + \delta) + \gamma\delta = \alpha^2 - \alpha(\alpha + \beta) + r$

$= -\alpha\beta + r = q + r$

Similarly $(\beta - \gamma)(\beta - \delta) = q + r$

So, ratio is 1.

Solution Sender of Maths Musing

SET-162

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BRAIN @ WORK

PROGRESSIONS

This article is a collection of shortcut methods, important formulas and MCQ's along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PET's.

SEQUENCE

A set of numbers forming a pattern is said to form a sequence. The individual numbers in sequence are called terms of the sequence. If the number of terms are finite then the sequence is called finite sequence. Otherwise it is an infinite sequence. The type of sequence depends upon the nature of the pattern.

A sequence is a function of natural numbers as domain with co-domain as the set of Real numbers (complex numbers). If Range is a subset of real numbers (complex numbers) then it is called a real sequence (complex sequence).

PROGRESSIONS

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the n^{th} term. Those sequences whose terms follow certain patterns are called progressions.

Series

By adding the terms of a sequence, we get a series *e.g.* $t_1 + t_2 + t_3 + t_4 + ...$

Note :

- For any sequence $t_n = S_n - S_{n-1}$, $n > 1$ where t_n is n^{th} term of the sequence and S_n is the sum of first n terms of the sequence and S_{n-1} is the sum of first $(n - 1)$ terms.
- m^{th} term from last in the sequence containing 'n' number of terms is equal to the $(n - m + 1)^{th}$ term from the beginning (where $m < n$).

ARITHMETIC SEQUENCE OR ARITHMETIC PROGRESSION (A.P.)

An Arithmetic progression (A.P.) is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference of the A.P.

If sequence $\{t_1, t_2, t_3, ...\}$ is such that $t_{n+1} - t_n$ = Fixed number (constant) $\forall\ n \in N$, then it is an A.P.

$\Rightarrow$ In an A.P ; $t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = ... t_{n-1} - t_{n-2} = t_n - t_{n-1} = d$

Now, if a is the first term and d the common difference, then generally A.P. can be written as

$a, a + d, a + 2d, a + 3d ....., a + (n - 1) d, a + nd, ...$

General Term or n^{th} Term of an A.P.

Let a be the first term, l be the last term (n^{th} term) and d be the common difference of a certain A.P. then its general term is given as $t_r = a + (r - 1)d$.

If the number of terms of an A.P. is n and value of last term is l, then $l = t_n = a + (n - 1)d$.

(i) n^{th} term of this A.P. from last $= t_n' = l - (n - 1)d$

(ii) If n^{th} term of any sequence is a linear expression in 'n', then the sequence is always an A.P. and coefficient of n is the common difference of A.P. *e.g.* if $t_n = An + B$, where A and B are constants is the n^{th} term of an A.P. whose common difference is A.

Sum of First n Terms of an A.P.

Let a be the first term, n be the number of terms, l be the last term (n^{th} term) and d be the common difference of the A.P., then the sum of n terms, S_n is given as,

$$S_n = \frac{n}{2}\{a + l\} \text{ or } S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$\text{or} \quad S_n = \frac{n}{2}\{2l - (n-1)d\}$$

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Note :

- If S_n denotes the sum of the first n terms of an A.P. and d is the common difference, then $d = S_n - 2S_{n-1} + S_{n-2}$.
- If the sum of 'n' terms of any sequence is a quadratic in 'n', then the sequence is an A.P. and its common difference is twice the coefficient of n^2. *e.g.* $S_n = 3n^2 + 4n$ is an A.P., with common difference 6.

Some Facts About A.P.

(i) The common difference of an A.P. can be zero, positive or negative.

(ii) Equal numbers, zero or non-zero are in A.P.

(iii) If each term of an A.P. is increased, decreased, multiplied and divided by the same quantity, then the resulting series will also be an A.P.

(iv) If $x_1, x_2, x_3, \ldots, x_n$ are in A.P., then

(a) Sum of the two terms equidistant from the beginning and end is a constant and is equal to the sum of first term and last term. *i.e.*, $x_1 + x_n = x_2 + x_{n-1} = x_3 + x_{n-2}$ and so on

(b) Any term of an A.P. (except the first and the last term) is half the sum of two terms equidistant from it. *i.e.*,

$$x_r = \frac{x_{r-k} + x_{r+k}}{2} \forall k, 0 \le k \le n - r$$

Selection of Terms of an A.P.

Number of Terms	Sequence
3	$a - d, a, a + d$
4	$a - 3d, a - d, a + d, a + 3d$
5	$a - 2d, a - d, a, a + d, a + 2d$

ARITHMETIC MEAN

If three terms are in A.P., then the middle term is called the Arithmetic mean (A.M.) between the other two, *i.e.*, if a, b, c are in A.P., then b is the A.M. of a and c.

(i) Single A.M. of n positive numbers

Let $a_1, a_2, a_3, \ldots, a_n$ be in A.P. and A be the A.M. of these numbers, then $A = \dfrac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$

In particular, let a and b be two given numbers and A be the A.M. between them. Then a, A, b are in A.P. and $A = \dfrac{a+b}{2}$

(ii) n-Arithmetic means between two numbers

Let a and b be two given numbers and $A_1, A_2, A_3, \ldots, A_n$ are n A.M.'s between them. Then $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in A.P.

Now $b = (n + 2)^{\text{th}}$ term *i.e.*, $b = a + (n + 2 - 1) d$

$$\therefore \quad d = \left(\frac{b-a}{n+1}\right)$$

Also, $A_1 = a + d, A_2 = a + 2d, \ldots, A_n = a + nd$

$$\Rightarrow \quad A_1 = a + \left(\frac{b-a}{n+1}\right), \ldots, A_n = a + n\left(\frac{b-a}{n+1}\right)$$

Note : The sum of n A.M.s, between a and b is equal to n times the single A.M. between a and b.

i.e. $A_1 + A_2 + A_3 + \ldots + A_n = \dfrac{n(a+b)}{2}$

GEOMETRIC PROGRESSION

A Geometric Progression (G.P.) is the sequence of numbers, whose first term is non-zero and each of the next term is obtained by multiplying its just preceding term by a non- zero constant quantity. This constant quantity is called the common ratio of the G.P.

Consider $t_1, t_2, t_3, \ldots, t_{n-1}, t_n, t_{n+1} \ldots$ be any sequence such that $\dfrac{t_2}{t_1} = \dfrac{t_3}{t_2} = \ldots = \dfrac{t_n}{t_{n-1}} = \dfrac{t_{n+1}}{t_n} = r$ then the sequence is said to be a G.P. and r is the common ratio of the G.P.

If a is the first term and r is the common ratio then G.P. can be written as

$a, ar, ar^2, \ldots, ar^{n-2}, ar^{n-1}, \ldots (a \ne 0)$

General Term or n^{th} Term of a G.P.

Let a be is the first term, r be the common ratio and l be the last term of a G.P., then

(i) n^{th} term is $t_n = ar^{n-1}$

(ii) n^{th} term from end $t_n' = \dfrac{l}{r^{n-1}} = lr^{1-n}$

Sum of First n Terms of a G.P.

Let a be the first terms, r be the common ratio and l be the last term of a G.P., then

(i) Sum of the first n terms is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{(r-1)}, & r > 1 \\[2ex] \dfrac{a(1 - r^n)}{1 - r}, & r < 1 \end{cases}$$

Also, $S_n = \dfrac{lr - a}{r - 1}$

(ii) Sum of an infinite G.P. is defined only when $|r| < 1$ when $n \to \infty$, $r^n \to 0$.

$$\therefore \quad S_{n \to \infty} = \frac{a}{1-r} \quad (\text{where } |r| < 1)$$

Properties of G.P.

(i) If each term of a G.P. is multiplied or divided by the same non-zero quantity then the resulting series will also be in G.P. but if each term of the series is increased or decreased by the same quantity, the resulting series is not a G.P. in general.

(ii) Equal non-zero numbers are in G.P.

(iii) If $x_1, x_2, x_3, \ldots\ldots$ are in G.P. ($x_i > 0 \ \forall\ i$), then $\log_m x_1, \log_m x_2, \log_m x_3, \ldots\ldots$ are in A.P. In this case the converse also holds good.

(iv) If $x_1, x_2, x_3, \ldots\ldots, x_n$ are in G.P., then the product of the terms equidistant from beginning and from end is same and equal to the product of the first and last term. It mean that

$x_1x_n = x_2\, x_{n-1} = x_3\, x_{n-2} = \ldots\ldots = x_r x_{n-(r-1)}$.

(v) Product of two terms that are equidistant from a given term is equal to the square of that term.

$x_r^2 = x_{r-k}x_{r+k}\ \forall\ k, 0 \le k < n - r$

- In a G.P., having an odd number of terms, the product of equidistant terms from beginning and end is equal to the square of the middle term.
- In a G.P. having an even number of terms, the product of equidistant terms from beginning and end is always equal to the product of two middle terms.

Selection of Terms in a G.P.

Number of Terms	Sequence
3	$a/r, a, ar$
4	$a/r^3, a/r, ar, ar^3$
5	$a/r^2, a/r, a, ar, ar^2$

GEOMETRIC MEAN

If three terms are in G.P., then the middle term is called the Geometric mean (G.M.) between the other two, so if a, b, c are in G.P., then b is the G.M. of a and c.

(i) Single G.M. of n positive numbers

Let $a_1, a_2, a_3, \ldots, a_n$ be in G.P. and G be the G.M. of these numbers, then $G = (a_1\, a_2\, a_3 \ldots\ldots a_n)^{1/n}$.

In particular

Let a and b be two numbers and G be the G.M. between them. Then a, G, b are in G.P.

$\therefore\quad G=\sqrt{ab};\ a>0, b>0$ and $G=-\sqrt{ab};$ if $a<0, b<0$

(ii) n-Geometric means between two numbers

Let a and b be two given numbers and $G_1, G_2, G_3, \ldots, G_n$ be n G.M.'s between them. Then $a, G_1, G_2, G_3, \ldots, G_n, b$ will be in G.P.

Now $b = (n + 2)^{\text{th}}$ term,

then $b = ar^{n+2-1} \Rightarrow\ r=\left(\dfrac{b}{a}\right)^{1/(n+1)}$

Also, $G_1 = ar, G_2 = ar^2, \ldots\ldots, G_n = ar^n$

$$\Rightarrow\quad G_1=a\left(\frac{b}{a}\right)^{1/(n+1)}, \ldots\ldots, G_n=a\left(\frac{b}{a}\right)^{n/(n+1)}$$

Note : The product of n geometric means between two given numbers a and b is equal to n^{th} power of the single geometric mean between a and b, where a and b are of the same sign.

i.e. $G_1 \cdot G_2 \cdot G_3 \ldots\ldots G_n=\left(\sqrt{ab}\right)^n = (G)^n$, where G is the geometric mean between a and b and $G_1, G_2, \ldots\ldots, G_n$ are n geometric means between a and b.

HARMONIC PROGRESSION

A sequence $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ of non-zero numbers is called a Harmonic progression, if the sequence $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots\ldots, \dfrac{1}{a_n}, \ldots\ldots$ is an Arithmetic progression.

Note :

(i) There is no general formula for the sum of any number of quantities in Harmonic Progression. Questions based on H.P. are generally solved by inverting the terms, and making use of the properties of the corresponding A.P.

(ii) If $x_1, x_2, x_3, \ldots\ldots, x_n$ are in H.P., then $\dfrac{1}{x_1}, \dfrac{1}{x_2}, \dfrac{1}{x_3}, \ldots\ldots, \dfrac{1}{x_n}$, are in A.P.

(a) n^{th} term of this H.P.,

$$T_n=\frac{x_1x_2}{x_2+(n-1)(x_1-x_2)}$$

(b) n^{th} term of this H.P. from end,

$$T_n'=\frac{x_1x_2x_n}{x_1x_2-x_n(n-1)(x_1-x_2)}$$

(iii) No term of a H.P. can be zero.

HARMONIC MEAN

If three terms are in H.P., then middle term is called the Harmonic mean (H.M.) between the other two, so if a, b, c are in H.P., then b is the H.M. of a and c.

(i) Single H.M. of n positive numbers

Let $a_1, a_2, a_3, \ldots\ldots, a_n$ be in H.P. and H be the H.M. of these numbers, then $H=\dfrac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots\ldots+\dfrac{1}{a_n}}$.

In particular, let a and b be two given numbers and H be the H.M. between them. Then a, H, b are in H.P., $H = \dfrac{2}{\frac{1}{a}+\frac{1}{b}} = \dfrac{2ab}{a+b}$

(ii) *n*-Harmonic means between two numbers

Let a and b be two given numbers and H_1, H_2, H_3, ..., H_n are 'n' H.M.'s between them. Then a, H_1, H_2, H_3, ..., H_n, b will be in H.P.

$$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \ldots\ldots, \frac{1}{H_n}, \frac{1}{b} \text{ are in A.P.}$$

If d is the common difference of the corresponding A.P. then $\frac{1}{b}$ is $(n + 2)^{\text{th}}$ term of the A.P.

$$\Rightarrow \frac{1}{b} = \frac{1}{a} + (n+2-1)d \Rightarrow d = \frac{(a-b)}{(n+1)ab}$$

$$\therefore \frac{1}{H_1} = \frac{1}{a} + d, \frac{1}{H_2} = \frac{1}{a} + 2d, \ldots\ldots, \frac{1}{H_n} = \frac{1}{a} + nd$$

$$\Rightarrow \frac{1}{H_1} = \frac{1}{a} + \frac{(a-b)}{(n+1)ab}, \frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab}, \ldots\ldots,$$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

Relationship Among A.M., G.M. and H.M.

(i) $A \geq G \geq H$ (if all numbers are +ve)

(ii) $A \leq G \leq H$ (if all numbers are -ve)

In both the cases equality holds only when the numbers are equal.

(iii) If A, G, H are respectively A.M., G.M., H.M. between two given numbers a, b then A, G, H form a G.P., *i.e.* $G^2 = AH$.

(iv) If A & G are A.M. & G.M. between two given numbers a, b then equation having a and b as its roots given as $x^2 - 2Ax + G^2 = 0$

(v) If A, G, H are A.M., G.M., H.M. between three given numbers a, b and c then equation having a, b, c as its roots is $x^3 - 3Ax^2 + \dfrac{3G^3}{H}x - G^3 = 0$.

(vi) If A and G be the arithmetic and the geometric means between two quantities, then the quantities are given by $A \pm \sqrt{(A+G)(A-G)}$.

SOME THEOREMS ON INEQUALITIES

Theorem 1 : The arithmetic mean of two or more positive quantities is greater than or equal to their geometric mean and Geometric mean is greater than or equal to their harmonic mean.

Theorem 2 : If a_1, a_2, a_3, ..., a_n are n positive numbers then

$$\frac{a_1 + a_2 + \ldots.. + a_n}{n} \geq (a_1 a_2 \ldots.. a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots.. + \frac{1}{a_n}}$$

Weighted Mean

Let a_1, a_2, ..., a_n be n positive real numbers and having frequency (weight) m_1, m_2, ..., m_n be n positive rational numbers. Then we define weighted Arithmetic mean (A_w), weighted Geometric mean (G_w) and weighted Harmonic mean (H_w) as

$$A_w = \frac{m_1 a_1 + m_2 a_2 + m_3 a_3 + \ldots.. + m_n a_n}{m_1 + m_2 + m_3 + \ldots.. + m_n};$$

$$G_w = (a_1^{m_1} \cdot a_2^{m_2} \cdot a_3^{m_3} \ldots.. \cdot a_n^{m_n})^{\frac{1}{(m_1+m_2+\cdots+m_n)}};$$

$$\text{and } H_w = \frac{m_1 + m_2 + m_3 + \ldots.. + m_n}{\left(\frac{m_1}{a_1} + \frac{m_2}{a_2} + \frac{m_3}{a_3} + \ldots.. + \frac{m_n}{a_n}\right)}$$

Arithmetic Mean of m^{th} Power

Let a_1, a_2, ..., a_n be n positive real numbers (not all equal) and let m be a real number, then

$$\frac{a_1^m + a_2^m + \ldots.. + a_n^m}{n} \geq \left(\frac{a_1 + a_2 + \ldots.. + a_n}{n}\right)^m; \text{ if } m \in R - [0, 1]$$

and equality hold only when $a_1 = a_2 = a_3 = \ldots.. a_n$

$$\frac{a_1^m + a_2^m + \ldots.. + a_n^m}{n} \leq \left(\frac{a_1 + a_2 + \ldots.. + a_n}{n}\right)^m; \text{if } m \in (0, 1) \text{ and}$$

equality hold only when $a_1 = a_2 = a_3 = \ldots.. a_n$

$$\frac{a_1^m + a_2^m + \ldots.. + a_n^m}{n} = \left(\frac{a_1 + a_2 + \ldots. + a_n}{n}\right)^m; \text{ if } m \in \{0, 1\}$$

PROBLEMS

1. If the third term of a G.P. is equal to 4, the product of its first five terms is equal to

(a) 2^6 (b) 2^{10}

(c) 2^8 (d) None of thess

2. If $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\theta$,

$z = \sum_{n=0}^{\infty} \cos^{2n}\theta . \sin^{2n}\theta$, where $\theta \in (0, \pi/2)$ then

(a) $xy = z$ (b) $xyz = xy + z$

(c) $xyz = yz + x$ (d) $xyz = xz + y$

3. If $(p + q)^{\text{th}}$ term of a G.P. is 'a' and its $(p - q)^{\text{th}}$ terms is 'b' where $a, b \in R^+$ then its p^{th} term is

(a) $\sqrt{\frac{a^3}{b}}$ (b) $\sqrt{\frac{b^3}{a}}$

(c) $\sqrt{ab}$ (d) None of these

4. If a, b, c are in H.P., then $\frac{a-b}{b-c}$ is always equal to

(a) c/a (b) a/c (c) b/a (d) a/b

5. If a, b, c are in A.P. , then the value of the expression $a^3 + c^3 - 8b^3$ is also equal to

(a) $2abc$ (b) $6abc$
(c) $4\ abc$ (d) None of these

6. If a, b, c, d are in H.P., then $ab + bc + cd$ is equal to

(a) $a(2c + d)$ (b) $c(2a + d)$
(c) $b(2c + d)$ (d) none of these

7. $A_i\ (x_i, y_i)$, $i = 1, 2, \ldots, n$ is a point on the curve $y_i = 2\sqrt{x_i}$. If $x_1, x_2, \ldots, x_n$ are in G.P. with $x_1 = 1$, $x_2 = 2$ then y_n is equal to

(a) $(\sqrt{2})^n$ (b) $(\sqrt{2})^{n+1}$
(c) $2n$ (d) none of these

8. If S_n denotes the sum of first 'n' terms of an A.P. , then $\frac{S_{3n} - S_{n-1}}{S_{2n} - S_{2n-1}}$ is always equal to

(a) $2n$ (b) $2n + 1$
(c) $2n - 1$ (d) none of these

9. If $4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8zx = 0$ then x, y, z are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

10. If $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$ then x, y, z are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

11. Sides a, b, c of a triangle are in G.P. If $\ln\left(\frac{5c}{a}\right)$, $\ln\left(\frac{3b}{5c}\right)$ and $\ln\left(\frac{a}{3b}\right)$ are in A.P., then triangle must be

(a) Isosceles (b) Equilateral
(c) Obtuse angled (d) none of these

12. If $(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{128}) = \sum_{r=0}^{n} x^r$ then 'n' is equal to

(a) 256 (b) 255
(c) 254 (d) none of these

13. If a, b, c are in H.P. where a, b, c are the sides of the triangle ABC, then

(a) Triangle is equilateral
(b) Triangle is isosceles
(c) $\cos B < 0$
(d) none of these

14. If a, b, c be positive real numbers forming a H.P., then $\frac{1}{b-a} + \frac{1}{b-c}$ is always equal to

(a) $2/a$ (b) $2/b$
(c) $2/c$ (d) none of these

15. If A_1 be the A.M. and G_1, G_2 be two G.Ms between two positive numbers a and b then $\frac{G_1^3 + G_2^3}{G_1G_2A_1}$ is equal to

(a) 2 (b) 1
(c) 5 (d) none of these

16. $\{b_i\}$, $i = 1, 2, \ldots n$ is arithmetic sequence. If $b_1 + b_5 + b_{10} + b_{15} + b_{20} + b_{24} = 255$, then $\sum_{i=1}^{24} b_i$ is equal to

(a) 600 (b) 900
(c) 300 (d) none of these

17. If $|x|, |x - 1|, |x + 1|$ are first three terms of an A.P. then sum of its first 10 terms is equal to

(a) 20 (b) 25 (c) 30 (d) 15

18. The sides a, b, c of a triangle ABC ; are in G.P. If 'r' be the common ratio of this G.P., then

(a) $r \in \left(\frac{\sqrt{5}-1}{2}, \infty\right)$ (b) $r \in \left(\frac{\sqrt{5}+1}{2}, \infty\right)$
(c) $r \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$ (d) $r \in \left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+3}{2}\right)$

19. If $a, b, c, d \in R^+$ and a, b, c, d are in H.P., then

(a) $a + d > b + c$ (b) $a + b > c + d$
(c) $a + c > b + d$ (d) None of these

20. Number of positive real values of x, such that x, $[x]$ and $\{x\}$ are in A.P. where [.] denotes the greatest integer function and {.} denotes fractional part, is equal to

(a) 1 (b) 2
(c) 0 (d) none of these

21. $\angle A, \angle B, \angle C$ of a triangle ABC are in A.P. If a, b, c are the corresponding sides, then

(a) $b^2 + c^2 - bc = a^2$ (b) $a^2 + c^2 - ac = b^2$
(c) $b^2 + a^2 - ab = c^2$ (d) None of these

22. Total number of positive real values of x such that x, $[x]$, $\{x\}$ are in H.P. where [.] denotes the greatest integer function and {.} denotes fraction part, is equal to

(a) Zero (b) 1
(c) 2 (d) none of these

23. If the first and $(2n + 1)^{th}$ term of an A.P., a G.P. and a H.P. (Consisting of positive terms) are equal and their $(n + 1)^{th}$ terms are a, b and c respectively then

(a) a, b, c are in A.P. (b) a, b, c are in G.P.
(c) a, b, c are in H.P. (d) none of these

24. If $\frac{1}{a}+\frac{1}{a-b}+\frac{1}{c}+\frac{1}{c-b}=0$ and $a + c \neq b$, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

25. If $a, b, c \in R^+ \sim \{1\}$ and the numbers $\log_a 100$, $2\log_b 10$, $2\log_c 5 + \log_c 4$ are in H.P. then

(a) $2b = a + c$ (b) $b^2 = ac$
(c) $b(a + c) = 2ac$ (d) none of these

26. If sides of a triangle are in A.P. and the greatest angle is two times the smallest angle in the triangle, then cosine of least angle is

(a) 3/8 (b) 2/3 (c) 1/8 (d) 3/4

27. a, b, c are distinct real numbers such that a, b, c are in A.P. and a^2, b^2, c^2 are in H.P. then

(a) $2b^2 = -ac$ (b) $4b^2 = -ac$
(c) $2b^2 = ac$ (d) $4b^2 = ac$

28. If $x_1, x_2, \ldots, x_n$ are in H.P. then $\sum_{r=1}^{n-1} x_r x_{r+1}$ is equal to

(a) $(n - 1)\, x_1 x_n$ (b) $n x_1\, x_n$
(c) $(n + 1)\, x_1 x_n$ (d) none of these

29. If a, b, c are in A.P., a, b, d are in G.P. then a, $a - b$, $d - c$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

30. If a, b, c are in H.P. then the value of $\frac{b+a}{b-a}+\frac{b+c}{b-c}$ is always equal to

(a) 1 (b) 2
(c) 3 (d) none of these

31. $\sum_{r=1}^{n}(a^r + br),\ a, b \in R^+$, is equal to

(a) $\frac{1-a^n}{1-a}+\frac{bn(n+1)}{2}$ (b) $\frac{b(1-a^n)}{1-a}+\frac{an(n+1)}{2}$
(c) $\frac{a(1-a^n)}{1-a}+\frac{bn(n+1)}{2}$ (d) None of these

32. $\sum_{r=1}^{n} r(n-r)$ is equal to

(a) $\frac{1}{6}n(n-1)(2n+1)$ (b) $\left(\frac{n(n+1)}{2}\right)^2$
(c) $\frac{n^2(n+1)}{6}$ (d) none of these

33. Value of $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots$ 'n' terms is equal to

(a) $\frac{4n^2+8n}{12(2n+1)(2n+3)}$
(b) $\frac{1}{4}\left(\frac{1}{(2n+1)(2n+3)}-\frac{1}{3}\right)$
(c) $\frac{4n^2+6n}{12(2n+1)(2n+3)}$ (d) none of these

34. If $a, b, c \in R^+$ such that $a + b + c = 18$ then the maximum value of $a^2b^3c^4$ is equal to

(a) $2^{18}.3^2$ (b) $2^{18}.3^3$
(c) $2^{19}.3^2$ (d) $2^{19}.3^3$

35. If $a, b, \in R^+$ then $\frac{bc}{b+c}+\frac{ac}{a+c}+\frac{ab}{a+b}$ is always

(a) $\leq\frac{1}{2}(a+b+c)$ (b) $\geq\frac{1}{3}\sqrt{abc}$
(c) $\leq\frac{1}{3}(a+b+c)$ (d) $\geq\frac{1}{2}\sqrt{abc}$

36. $\sum_{r=1}^{n}\frac{r}{1.3.5.7.9.....(2r+1)}$ is equal to

(a) $\frac{1}{2}\left(1-\frac{1}{1.3.5.....(2n+1)}\right)$
(b) $\frac{1}{4}\left(1-\frac{1}{1.3.5.....(2n+1)}\right)$
(c) $\frac{1}{3}\left(1-\frac{1}{1.3.5.....(2n+1)}\right)$
(d) None of these

37. If a, b, c are distinct positive real numbers such that $a^2 + b^2 + c^2 = 1$ then $ab + bc + ca$ is always

(a) <1 (b) >1
(c) $=1$ (d) none of these

38. If $a, b, c, \in R^+$ then $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ is always

(a) ≥ 12 (b) ≥ 9
(c) ≤ 12 (d) none of these

39. H_1 is the H.M. and G_1 the G.M. of positive real numbers 'a' and 'b'. If $H_1 : G_1 = 4 : 5$ then $a : b$ is

(a) 5 : 4 (b) 4 : 1
(c) 1 : 5 (d) none of these

40. If $x_n > 1 \ \forall \ n \in N$, then the minimum value of $\log_{x_2} x_1 + \log_{x_3} x_2 + \log_{x_4} x_3 + ... + \log_{x_n} x_{n-1} + \log_{x_1} x_n$ is equal to

(a) n (b) $2n$

(c) $n + 1$ (d) none of these

SOLUTIONS

1. (b): Let the first term be 'a' and common ratio be 'r'

$\Rightarrow \ 4 = ar^2$

Required product $= a \,.\, ar \,.\, ar^2 \,.\, ar^3 \,.\, ar^4$

$= a^5 \,.\, r^{10} = (ar^2)^5 = 4^5$

2. (b): $x = \sum_{n=0}^{\infty} \cos^{2n}\theta = \dfrac{1}{1-\cos^2\theta} = \operatorname{cosec}^2\theta$

$y = \sum_{n=0}^{n} \sin^{2n}\theta = \dfrac{1}{1-\sin^2\theta} = \sec^2\theta$

$z = \sum_{n=0}^{\infty} \sin^{2n}\theta.\cos^{2n}\theta = \dfrac{1}{1-\sin^2\theta.\cos^2\theta} = \dfrac{1}{1-\frac{1}{x}.\frac{1}{y}}$

$\Rightarrow \ xyz - z = xy \ \Rightarrow \ xyz = xy + z$

3. (c): Let 'A' be first term and 'r' be the common ratio. Then

$a = Ar^{p+q-1}, \ b = A.r^{p-q-1}$

$\Rightarrow \ ab = A^2.\ r^{2p-2}$

$\Rightarrow \ \sqrt{ab} = A.r^{p-1} = p^{\text{th}}$ term

4. (b): $\because \ a,b,c \in$ H.P. $\therefore \dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b}$

$\Rightarrow \ \dfrac{a-b}{ab} = \dfrac{b-c}{bc} \ \Rightarrow \ \dfrac{a-b}{b-c} = \dfrac{a}{c}$

5. (d): $\because \ a, b, c, \in$ A.P. $\therefore 2b = a + c$

$\Rightarrow \ 8b^3 = (a + c)^3 = a^3 + c^3 + 3ac\,(a + c)$

$\Rightarrow \ 8b^3 = a^3 + c^3 + 3ac(2b)$

$\Rightarrow \ a^3 + c^3 - 8b^3 = -6abc$

6. (b): $ab + bc + cd = b(a + c) + cd = 2ac + cd = c(2a + d)$

7. (b): $y_n^2 = 4.x_n = 4.x_1.\left(\dfrac{x_2}{x_1}\right)^{n-1} = 4 \cdot 1 \cdot (2)^{n-1} = 2^{n+1}$

$\Rightarrow \ y_n = 2^{\frac{n+1}{2}} = (\sqrt{2})^{n+1}$

8. (b): $S_{3n} = \dfrac{3n}{2}(2a + (3n-1)d)$

$S_{n-1} = \dfrac{n-1}{2}(2a + (n-2)d)$

$\Rightarrow \ S_{3n} - S_{n-1}$

$= (a(3n - n + 1) + \dfrac{1}{2}d(3n(3n - 1) - (n - 1)(n - 2))$

$= a(2n + 1) + d(4n^2 - 1)$

$= (2n + 1)(a + (2n - 1)d)$

$S_{2n} - S_{2n-1} = T_{2n} = a + (2n - 1)d$

9. (c): We have, $4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8zx = 0$

$\Rightarrow \ 4x^2 - 2x(3y + 4z) + 9y^2 + 16z^2 - 12yz = 0$

Now, $4(3y + 4z)^2 - 16(9y^2 + 16z^2 - 12yz) \geq 0$

$\Rightarrow \ 9y^2 + 16z^2 \geq 36y^2 + 64z^2 - 48yz - 24yz$

$\Rightarrow \ 27y^2 + 48z^2 - 72yz \leq 0 \Rightarrow 3(9y^2 + 16z^2 - 24yz) \leq 0$

$\Rightarrow \ (3y - 4z)^2 \leq 0 \Rightarrow 3y = 4z$

Putting these values in the given equation, we get

$4x^2 + 18y^2 - 6xy - 9y^2 - 6xy = 0$

$\Rightarrow \ 4x^2 + 9y^2 - 12xy = 0 \Rightarrow (2x - 3y)^2 = 0 \Rightarrow 2x = 3y$

Thus, $2x = 3y = 4z = \lambda$ (say)

$\Rightarrow \ x = \dfrac{\lambda}{2}, \ y = \dfrac{\lambda}{3}, \ z = \dfrac{\lambda}{4}$

$\Rightarrow \ x, y, z$ are in H.P.

10. (c): $x^2 + 9y^2 + 25z^2 = 15yz + 5xz + 3xy$

$\Rightarrow \ x^2 - x(3y + 5z) + 9y^2 + 25z^2 - 15yz = 0$

Now, $(3y + 5z)^2 - 4(9y^2 + 25z^2 - 15yz) \geq 0$

$\Rightarrow \ 9y^2 + 25z^2 + 30yz - 36y^2 - 100z^2 + 60yz \geq 0$

$\Rightarrow \ 27y^2 + 75z^2 - 90yz \leq 0 \Rightarrow 9y^2 + 25z^2 - 30yz \leq 0$

$\Rightarrow \ (3y - 5z)^2 \leq 0 \ \Rightarrow \ 3y = 5z$

Putting these values in the given equation, we get

$x = 3y = 5z$

$\Rightarrow \ x = \dfrac{\lambda}{1}, \ y = \dfrac{\lambda}{3}, \ z = \dfrac{\lambda}{5}$

$\Rightarrow \ x, y, z$ are in H.P.

11. (c): We have, $b^2 = ac$ ...(1)

Also $2\ln\left(\dfrac{3b}{5c}\right) = \ln\left(\dfrac{5c}{a}\right) + \ln\left(\dfrac{a}{3b}\right)$

$\Rightarrow \ \dfrac{9b^2}{25c^2} = \dfrac{5c}{a}.\dfrac{a}{3b} \Rightarrow 27b^3 = 125c^3 \Rightarrow 3b = 5c$ ...(2)

Using (1) and (2), we get $a = \dfrac{b^2.5}{3b} = \dfrac{5}{3}b$

Now, $b^2 + c^2 - a^2 = \lambda^2 + \dfrac{9\lambda^2}{25} - \dfrac{25\lambda^2}{9}$

$= \left(\dfrac{225 + 81 - 625}{225}\right)\lambda^2 < 0$

$\Rightarrow \ \cos A < 0$

$\Rightarrow$ Triangle is obtuse angled.

12. (b): $(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) ..... (1 + x^{2^7})$

$= \dfrac{(1-x^2)(1+x^2)(1+x^4)(1+x^8).....(1+x^{2^7})}{(1-x)}$

$$=\frac{1}{(1-x)}(1-x^4)(1+x^4)\ .....\ (1+x^{2^7})$$

$$=\frac{1-x^{2^8}}{1-x} \Rightarrow n = 255$$

13. (d): We have, $b=\frac{2ac}{a+c}$

Now, $\cos B=\frac{a^2+c^2-b^2}{2ac}$

$$=\frac{a^2+c^2-\frac{4a^2c^2}{(a+c)^2}}{2ac}=\frac{(a+c)^2(a^2+c^2)-4a^2c^2}{2ac(a+c)^2}$$

$$=\frac{(a^2+c^2)^2+2ac(a^2+c^2)-4a^2c^2}{2ac(a+c)^2}$$

$$=\frac{(a^2-c^2)^2+2ac(a^2+c^2)}{2ac(a+c)^2}>0$$

14. (b): $\frac{1}{b-a}+\frac{1}{b-c}=\frac{2b-(a+c)}{(b-a)(b-c)}$

$$=\frac{2b-(a+c)}{b^2-b(a+c)+ac}=\frac{2b-\frac{2ac}{b}}{b^2-2ac+ac}=\frac{2}{b}\ (\because a, b, c, \in \text{H.P.})$$

15. (a): $A_1=\frac{a+b}{2}, G_1=a\left(\frac{b}{a}\right)^{1/3}, G_2=a.\left(\frac{b}{a}\right)^{2/3}$

$G_1^3=a^2b$, $G_2^3=b^2a$, $G_1G_2=a^2.\left(\frac{b}{a}\right)=ab$

Now, $\frac{G_1^3+G_2^3}{G_1G_2A_1}=\frac{ab(a+b)\cdot 2}{ab\cdot(a+b)}=2$

16. (b): $b_1+b_5+b_{10}+b_{15}+b_{20}+b_{24}$

$=6b_1+(4d+9d+14d+19d+23d)\Rightarrow 225=3(2b_1+23d)$

Now, $\sum_{i=1}^{24} b_1=\frac{24}{2}(2b_1+23d)=12\times 75=900$

17. (b) : We have, $2|x-1|=|x|+|x+1|$

Let $x\le -1$

$\Rightarrow\ 2(1-x)=-x-x-1$

$\Rightarrow\ 2=-1$ (not possible)

For $-1<x\le 0$, we get

$2(1-x)=-x+x+1$

$\Rightarrow\ x=\frac{1}{2}$ (not possible)

For $0<x\le 1$, we get

$2(1-x)=x+x+1$

$\Rightarrow\ x=\frac{1}{4}$

For $x>1$, we get

$2(x-1)=x+x+1$

$\Rightarrow\ -2=1$ (not possible).

Thus, $x=\frac{1}{4}$ and sequence is $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, ...$

$\therefore$ Required sum $=\frac{10}{2}\left(\frac{2}{4}+9.\left(\frac{3}{4}-\frac{1}{4}\right)\right)=25$

18. (c) : Let $b=ar$, $c=ar^2$

For triangle ABC,

$a+b>c$

$a+ar>ar^2$

$\Rightarrow\ r^2-r-1<0 \Rightarrow \frac{1-\sqrt{5}}{2}<r<\frac{1+\sqrt{5}}{2}$

Also, $b+c>a$

$\Rightarrow\ a(r+r^2)>a \Rightarrow r^2+r-1>0$

$\Rightarrow\ r\in\left(-\infty, \frac{-1-\sqrt{5}}{2}\right)\cup\left(\frac{-1+\sqrt{5}}{2}, \infty\right)$

Finally, $a+c>b$

$\Rightarrow\ a(1+r^2)>ar \Rightarrow r^2-r+1>0$

which is true for all values of r.

In this case 'r' can not be negative

$\Rightarrow\ r\in\left(\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2}\right)$

19. (a) : Using A.M. $\ge$ H.M.

we get $\frac{a+c}{2}>b \Rightarrow a+c>2b$

and $\frac{b+d}{2}>c \Rightarrow b+d>2c$

$\Rightarrow\ a+c+b+d>2b+2c \Rightarrow a+d>b+c$

20. (a) : We have,

$2[x]=x+\{x\}=[x]+2\{x\}$

$\Rightarrow\ \{x\}=\frac{[x]}{2} \Rightarrow 0\le\frac{[x]}{2}<1\Rightarrow 0\le[x]<2\Rightarrow [x]=0, 1$

For $[x]=0$, $\{x\}=0$

$\Rightarrow\ x=0$ (not possible $\because$ x is positive)

For $[x]=1$, $\{x\}=\frac{1}{2} \Rightarrow x=\frac{3}{2}$

Thus there is only one value of x.

21. (b): We have, $2\angle B=\angle A+\angle C$

$\Rightarrow\ \angle B=\frac{\pi}{3} \Rightarrow \cos\frac{\pi}{3}=\frac{a^2+c^2-b^2}{2ac}=\frac{1}{2}$

$\Rightarrow\ a^2+c^2-ac=b^2$

22. (b): We must have $[x]=\frac{2\{x\}x}{\{x\}+x}$

$\Rightarrow\ [x]\{x\}+[x]([x]+\{x\})=2\{x\}([x]+\{x\})$

$\Rightarrow [x]^2 = 2\{x\}^2 \Rightarrow \{x\}^2 = \dfrac{[x]^2}{2}$

$\Rightarrow 0 \le \dfrac{[x]^2}{2} < 1 \Rightarrow 0 \le [x]^2 < 2$

$\Rightarrow [x] = 0$ or $[x] = 1$

For $[x] = 0 \Rightarrow \{x\} = 0$

$\Rightarrow x = 0$ (not possible)

For $[x] = 1 \Rightarrow \{x\} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow x = 1 + \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}+1}{\sqrt{2}}$ is the only possible value of x.

23. (b): A.P. $\rightarrow$ first term $= a_1$, common difference $= d$

G.P. $\rightarrow$ first term $= a_1$, common ratio $= r$

H.P. $\rightarrow$ first term $= a_1$, common ratio of corresponding A.P. $= d_1$

We have, $a_1 + 2nd = a_1 r^{2n}$

$$= \frac{1}{\frac{1}{a_1} + 2nd_1} = \lambda \qquad \text{(say)}$$

$\Rightarrow 2nd = \lambda - a_1, r^n = \sqrt{\dfrac{\lambda}{a_1}}, 2nd_1 = -\dfrac{1}{a_1} + \dfrac{1}{\lambda}$

$\Rightarrow a = a_1 + nd = a_1 + \dfrac{\lambda - a_1}{2} = \dfrac{\lambda + a_1}{2}$

$b = a_1 r^n = a_1\sqrt{\dfrac{\lambda}{a_1}} = \sqrt{\lambda a_1}$

$c = \dfrac{1}{\frac{1}{a_1} + nd_1} = \dfrac{1}{\frac{1}{a_1} - \frac{1}{2a_1} + \frac{1}{2\lambda}} = \dfrac{2\lambda a_1}{\lambda + a_1}$

$\Rightarrow ac = \lambda\, a_1 = b^2 \Rightarrow a, b, c$ are in G.P.

24. (c) : We have,

$\Rightarrow \left(\dfrac{1}{a} + \dfrac{1}{c-b}\right) + \left(\dfrac{1}{c} + \dfrac{1}{a-b}\right) = 0$

$\Rightarrow \dfrac{a+c-b}{a(c-b)} + \dfrac{a+c-b}{c(a-b)} = 0 \Rightarrow b = \dfrac{2ac}{a+c}$

$\Rightarrow a, b, c$ are in H.P.

25. (b): $\log_a 100 = 2\log_a 10$

and $2\log_c 5 + \log_c 4 = \log_c 100 = 2\log_c 10$

Since $2\log_a 10, 2\log_b 10, 2\log_c 10$ are in H.P.

$\therefore \dfrac{2}{2\log_b 10} = \dfrac{1}{2\log_a 10} + \dfrac{1}{2\log_c 10}$

$\Rightarrow 2\log_{10} b = \log_{10} a + \log_{10} c$

$\Rightarrow b^2 = ac$

$\Rightarrow a, b, c$ are in G.P.

26. (d): Let the sides be $a_1 - d_1, a_1, a_1 + d_1$

$\therefore C = 2A$

$\Rightarrow \sin C = 2\sin A\,.\cos A$

Thus, $\dfrac{c}{2R} = 2.\dfrac{a}{2R}.\left(\dfrac{b^2+c^2-a^2}{2bc}\right)$

$\Rightarrow bc^2 = a(b^2 + c^2 - a^2)$

$\Rightarrow a_1(a_1 + d_1)^2 = (a_1 - d_1)(a_1^2 + (a_1 + d_1)^2 - (a_1 - d_1)^2)$

$\Rightarrow a_1(a_1 + d_1)^2 = (a_1 - d_1)(4a_1d_1 + a_1^2)$

$\Rightarrow a_1^2 + d_1^2 + 2a_1d_1 = 4a_1d_1 + a_1^2 - 4d_1^2 - a_1d_1$

$\Rightarrow 5d_1^2 = a_1d_1 \Rightarrow a_1 = 5d_1$

Now, $\cos A = \dfrac{\sin C}{2\sin A} = \dfrac{c}{2a} = \dfrac{a_1 + d_1}{2(a_1 - d_1)} = \dfrac{6d_1}{2(4d_1)} = \dfrac{3}{4}$

27. (a) : We have, $2b = a + b,\ b^2 = \dfrac{2a^2c^2}{a^2+c^2}$

$\Rightarrow \dfrac{(a+c)^2}{4} = \dfrac{2a^2c^2}{a^2+c^2}$

$\Rightarrow (a+c)^2 = \dfrac{8a^2c^2}{a^2+c^2} = \dfrac{8a^2c^2}{(a-c)^2+2ac}$

$\Rightarrow (a^2 - c^2)^2 + 2ac(a + c)^2 = 8a^2c^2$

$\Rightarrow (a - c)^2((a + c)^2 + 2ac) = 0 \Rightarrow (a + c)^2 + 2ac = 0$

$\Rightarrow 4b^2 + 2ac = 0$

28. (a) : Clearly, $\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots\ldots \dfrac{1}{x_n}$ will be in A.P.

$\Rightarrow \dfrac{1}{x_2} - \dfrac{1}{x_1} = \dfrac{1}{x_3} - \dfrac{1}{x_2} = \ldots\ldots \dfrac{1}{x_{r+1}} - \dfrac{1}{x_r} = \ldots\ldots = \lambda$ (say)

$\Rightarrow \displaystyle\sum_{r=1}^{n-1} x_r\, x_{r+1} = \frac{1}{\lambda}\sum_{r=1}^{n-1}(x_r - x_{r+1}) = \frac{1}{\lambda}(x_1 - x_n)$

Now, $\dfrac{1}{x_n} = \dfrac{1}{x_1} + (n-1)\lambda$

$\Rightarrow \dfrac{x_1 - x_n}{x_1x_n} = (n-1)\lambda$

$\Rightarrow \displaystyle\sum_{r=1}^{n-1} x_r x_{r+1} = (n-1)x_1x_n$

29. (b): $2b = a + c,\ b^2 = ad$

$(a - b)^2 = a^2 + b^2 - 2ab = a^2 + b^2 - a(a + c)$

$$= b^2 - ac = a\left(\frac{b^2}{a} - c\right) = a(d - c)$$

$\Rightarrow a, (a - b), (d - c)$ are in G.P.

30. (b): $b=\frac{2ac}{a+c} \Rightarrow b+a=\frac{2ac}{a+c}+a=\frac{a^2+3ac}{a+c}$

Now, $b-a=\frac{2ac}{a+c}-a=\frac{ac-a^2}{a+c}$

and $b+c=\frac{2ac}{a+c}+c=\frac{c^2+3ac}{a+c}$

and $b-c=\frac{2ac}{a+c}-c=\frac{ac-c^2}{a+c}$

$\Rightarrow \frac{b+a}{b-a}+\frac{b+c}{b-c}=\frac{a^2+3ac}{ac-a^2}+\frac{c^2+3ac}{ac-c^2}=\frac{a+3c}{c-a}+\frac{c+3a}{a-c}$

$=\frac{1}{c-a}(a+3c-c-3a)=2$

31. (c): $\sum_{r=1}^{n}(a^r+br)=\sum_{r=1}^{n}a^r+b\sum_{r=1}^{n}r$

$=\frac{a(1-a^n)}{1-a}+\frac{bn(n+1)}{2}$

32. (d) : $S=\sum_{r=1}^{n}r(n-r)=n\sum_{r=1}^{n}r-\sum_{r=1}^{n}r^2$

$=\frac{n^2(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)}{6}(3n-2n-1)$

$=\frac{n(n+1)(n-1)}{6}=\frac{n(n^2-1)}{6}$

33. (a): $T(r)=\frac{1}{(2r-1)(2r+1)(2r+3)}$

$=\frac{2r+3-(2r-1)}{4(2r-1)(2r+1)(2r+3)}$

$=\frac{1}{4}\left[\frac{1}{(2r-1)(2r+1)}-\frac{1}{(2r+1)(2r+3)}\right]$

$\Rightarrow \sum_{r=1}^{n}T(r)=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2n+1)(2n+3)}\right]$

34. (d): $a + b + c = 18$

$\Rightarrow 2.\frac{a}{2}+3.\frac{b}{3}+4.\frac{c}{4}=18$

Using weighted A.M. and G.M. inequality, we get

$\frac{2.\frac{a}{2}+3.\frac{b}{3}+4.\frac{c}{4}}{9}\ge\left(\left(\frac{a}{2}\right)^2\left(\frac{b}{3}\right)^3\left(\frac{c}{4}\right)^4\right)^{1/9}$

$\Rightarrow 2^9\ge\frac{a^2}{2^2}.\frac{b^3}{3^3}.\frac{c^4}{4^4} \Rightarrow a^2b^3c^4 \le 3^3\ .\ 2^{19}$

35. (a): Using A.M. and H.M. inequality, we get

$\frac{2bc}{b+c}\le\frac{b+c}{2},\ \frac{2ac}{a+c}\le\frac{a+c}{2},\ \frac{2ab}{a+b}\le\frac{a+b}{2}$

$\Rightarrow \frac{bc}{b+c}+\frac{ac}{a+c}+\frac{ab}{a+b}\le\frac{1}{2}(a+b+c)$

36.(a) : $T(r)=\frac{r}{1\cdot3\cdot5.....(2r+1)}$

$=\left(\frac{2r+1-1}{2(1\cdot3\cdot5\ .....\ (2r+1))}\right)$

$=\frac{1}{2}\left(\frac{1}{1\cdot3\cdot5\ .....\ (2r-1)}-\frac{1}{1\cdot3\cdot5\ .....\ (2r+1)}\right)$

$=-\frac{1}{2}[V(r)-V(r-1)]$

$\sum_{r=1}^{n}T(r)=-\frac{1}{2}(V(n)-V(0))$

$=\frac{1}{2}\left(1-\frac{1}{1\cdot3\cdot5.....(2n+1)}\right)$

37. (a): $\frac{a^2+b^2}{2}\ge ab, \frac{b^2+c^2}{2}\ge bc, \frac{c^2+a^2}{2}\ge ca$

$\Rightarrow a^2 + b^2 + c^2 > ab + bc + ca$

$\Rightarrow ab + bc + ca < 1$

38. (b): $\frac{a+b+c}{3}\ge\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ (A.M. $\ge$ H.M.)

$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9$

39. (b): $H_1=\frac{2ab}{a+b}, G_1=\sqrt{ab}$

$\Rightarrow \frac{H_1}{G_1}=\frac{2ab}{(a+b)\sqrt{ab}} \Rightarrow \frac{4}{5}=\frac{2\sqrt{ab}}{a+b}$

$\Rightarrow 9=\frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} \Rightarrow 3=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

$\Rightarrow a : b = 4 : 1$

40. (a): $\log_{x_2}x_1+\log_{x_3}x_2+\log_{x_4}x_3+.....$

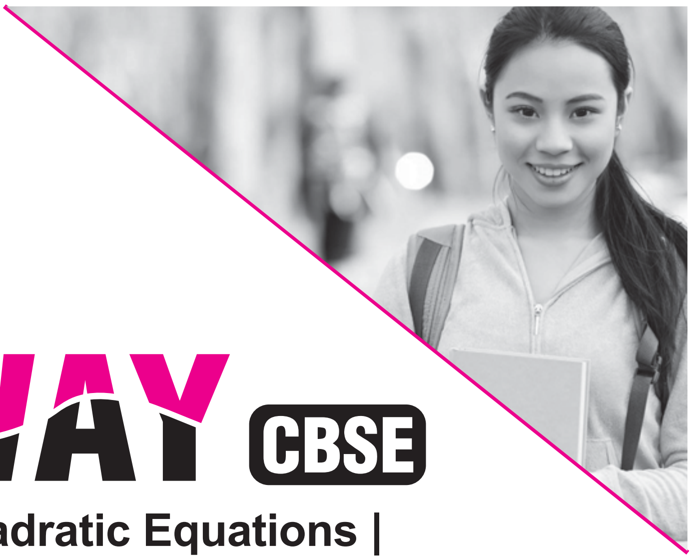
$.....+\log_{x_n}x_{n-1}+\log_{x_1}x_n$

$\frac{\log x_1}{\log x_2}+\frac{\log x_2}{\log x_3}+\frac{\log x_3}{\log x_4}+.......+\frac{\log x_{n-1}}{\log x_n}+\frac{\log x_n}{\log x_1}$

G.M. of these 'n' numbers in 1. Thus minimum value of the given expression is n.

CLASS XI | Series 4

ACE YOUR WAY CBSE



Complex Numbers and Quadratic Equations | Linear Inequalities

HIGHLIGHTS

COMPLEX NUMBERS

- A number of the form $a+ib\ \forall\ a,b\in R$ (*i.e.,* set of real numbers) is called complex number, where $i=\sqrt{-1}$.
- Complex number is generally denoted by z *i.e.,* $z = a + ib$ is a complex number whose real part is 'a' and imaginary part is 'b'.
- A complex number is said to be
  (i) Purely real iff $\text{Im}(z) = 0$
  (ii) Purely imaginary iff $\text{Re}(z) = 0$

EQUALITY OF TWO COMPLEX NUMBERS

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Then we say that $z_1 = z_2$ iff $a_1 = a_2$ and $b_1 = b_2$ or
$z_1 = z_2$ iff $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

ALGEBRA OF COMPLEX NUMBERS

	Definition	Properties
Addition	The sum of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ is given by $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$	**(i) Closure Law :** The sum of two complex numbers is a complex number. **(ii) Commutative Law :** Let z_1 and z_2 be two complex numbers. Then, $z_1 + z_2 = z_2 + z_1$ **(iii) Associative Law :** Let z_1, z_2 and z_3 be any three complex numbers. Then, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ **(iv) Existence of Identity :** Let $z = a + ib$ be a complex number, then there exists a complex number $0 = 0 + i0$ such that $z + 0 = z = 0 + z$, here $0 + i0$ is the additive identity. **(v) Existence of inverse :** For every complex number $z = a + ib$, there exists another complex number $-z = (-a) + i(-b)$ such that, $z + (-z) = 0 = (-z) + z$, here $-z = (-a) + i(-b)$ is the additive inverse of z.

Multiplication	The product of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ is given by $z_1z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$	**(i) Closure Law :** The product of two complex numbers is a complex number. **(ii) Commutative Law :** Let z_1 and z_2 be any two complex numbers. Then, $z_1z_2 = z_2z_1$ **(iii) Associative law :** Let z_1, z_2 and z_3 be any three complex numbers. Then, $(z_1z_2)z_3 = z_1(z_2z_3)$ **(iv) Distributive Law :** For any three complex numbers z_1, z_2 and z_3, we have $z_1 (z_2 + z_3) = z_1z_2 + z_1z_3$ and $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$ **(v) Existence of Identity :** For every complex number, $z = a + ib$, there exists another complex number $1 = 1 + i0$ such that $z \cdot 1 = z = 1 \cdot z$, where 1 is the multiplicative identity. **(vi) Existence of Inverse :** For every non-zero complex number, $z = a + ib$ $(a \neq 0, b \neq 0)$, there exists a complex number $\frac{1}{z} = \frac{a}{a^2+b^2} + i\left(\frac{-b}{a^2+b^2}\right)$ such that $z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z$, where $\frac{1}{z}$ is called the multiplicative inverse.

Note :

- Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ $(z_2 \neq 0)$, then
  $$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\left(\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}\right)$$
- $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$ and $i^{-4} = 1$
- $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$ and $i^{4k+3} = -i$ for any integer k.

SQUARE ROOT OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. Then

$$\sqrt{a+ib} = \begin{cases} \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right], & \text{when } b > 0 \\ \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right], & \text{when } b < 0 \end{cases}$$

MODULUS AND CONJUGATE OF A COMPLEX NUMBER

Let $z = a + ib$ be any complex number, then

- **Modulus** of z is denoted by $|z|$ and defined as $|z| = \sqrt{a^2 + b^2}$.
- **Conjugate** of z is denoted by $\bar{z}$ and defined as $\bar{z} = a - ib$.

For any two complex numbers z_1 and z_2, we have the following properties of modulus and conjugate :

(i) $|z_1z_2| = |z_1||z_2|$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, |z_2| \neq 0$

(iii) $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$

(iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

ARGUMENT OF A COMPLEX NUMBER

The argument or amplitude of a complex number $z = a + ib$ is the value of θ which satisfies the given two equations,

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

It is denoted by arg(z) or amp(z).

Note :

- Principal argument is the value of θ lying in the interval $(-\pi, \pi]$.
- General value of the argument is the value $(2n\pi + \theta)$, where θ is the principal argument.

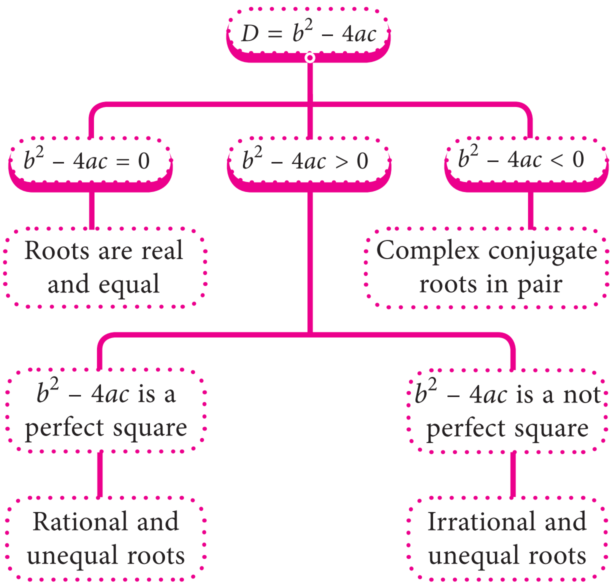
POLAR FORM OF A COMPLEX NUMBER

Every complex number $z = a + ib$ can be put in the form $r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$. This form is called the polar form of the complex number.

QUADRATIC EQUATIONS

An equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$ is known as quadratic equation. The solutions of the above defined equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$



LINEAR INEQUALITIES

DEFINITION

A statement involving the symbols '>', '<', '≥' or '≤' is called an inequality.

Some important points :

- Inequalities which do not involve variables are called numerical inequalities.
- Inequalities which involve variables are called literal inequalities.
- Inequalities involving the symbol '>' or '<' are called strict inequalities.
- Inequalities involving the symbols '≥' or '≤' are called slack inequalities.

ALGEBRAIC SOLUTIONS OF LINEAR INEQUALITIES IN ONE VARIABLE AND THEIR GRAPHICAL REPRESENTATION

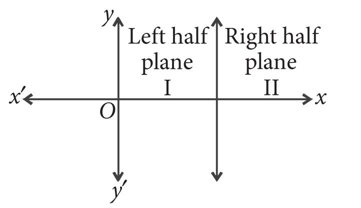
Any solution of an inequality in one variable is a value of the variable which makes it a true statement. We can state the following rules for solving an inequality :

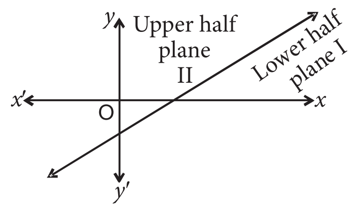
- **Rule 1 :** Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of inequality.
- **Rule 2 :** (i) If both sides of an inequality are multiplied (or divided) by the same positive number, then sign of inequality remains unchanged.

  (ii) If both sides are multiplied (or divided) by a negative number, then the sign of inequality is reversed.

GRAPHICAL SOLUTION OF LINEAR INEQUALITIES IN TWO VARIABLES

- A line divides the Cartesian plane into two parts. Each part is known as a half-plane. A vertical line will divide the plane in left and right half planes and a non-vertical line will divide the plane into lower and upper half planes.





- The region containing all the solutions of an inequality is called the solution region.
- In order to identify the half plane represented by an inequality, it is just sufficient to take any point (a, b) not on the line and check whether it satisfies the inequality or not. If it satisfies, then the inequality represents the half plane which contains the point, otherwise the inequality represents the half plane which does not contain the point.

- If an inequality is of type $ax + by \geq c$ or $ax + by \leq c$, then the point on the line $ax + by = c$ are also included in the solution region. So we draw a dark line in the solution region.
- If an inequality is of the form $ax + by > c$ or $ax + by < c$, then the points on the line $ax + by = c$ are not to be included in the solution region. So we draw a broken or dotted line in the solution region.

PROBLEMS

Very Short Answer Type

1. Simplify : $i^{n+100} + i^{n+50} + i^{n+48} + i^{n+46}$
2. Solve : $x^2 + 2 = 0$
3. Solve the inequation, $-12 < 3x - 5 \leq -4$.
4. If z_1 and z_2 are $1 - i$ and $-2 + 4i$ respectively, find $\text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$.
5. Write the real and imaginary parts of the following complex number $\sqrt{37} + \sqrt{-19}$.

Long Answer Type - I

6. Find the real θ such that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real.
7. Find the conjugate and argument of $\frac{2+i}{4i+(1+i)^2}$.
8. Solve : $x^2 - 14x + 58 = 0$
9. If $|z-2| = 2|z-1|$, where z is a complex number, show that $|z|^2 = \frac{4}{3}\text{Re}(z)$.
10. Find the conjugate, modulus and principal argument of $\sqrt{2} - \sqrt{2}i$.

Long Answer Type - II

11. Evaluate : $\sqrt{4+3\sqrt{-20}} + \sqrt{4-3\sqrt{-20}}$
12. If $\frac{3}{2+\cos\theta+i\sin\theta} = a + ib$, prove that $a^2 + b^2 = 4a - 3$.
13. Solve $\frac{5}{x-2} > 3$ and represent the solution set on the number line.
14. Express $(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)$ as the sum of two squares.
15. Solve the following simultaneous linear inequations : $x + 2y \leq 10$, $x + y \leq 6$, $x \leq 4$, $x \geq 0$ and $y \geq 0$.

SOLUTIONS

1. We have,

$$i^{n+100} + i^{n+50} + i^{n+48} + i^{n+46}$$
$$= i^n(i^{100} + i^{50} + i^{48} + i^{46})$$
$$= i^n[(i^2)^{50} + (i^2)^{25} + (i^2)^{24} + (i^2)^{23}]$$
$$= i^n[(-1)^{50} + (-1)^{25} + (-1)^{24} + (-1)^{23}]$$
$$= i^n[1 - 1 + 1 - 1] = i^n \cdot 0 = 0$$

2. We have, $x^2 + 2 = 0$

$$\Rightarrow x^2 = -2 \Rightarrow x = \pm\sqrt{-2} = \pm i\sqrt{2}$$
$$\therefore \text{ Solution set } = \{i\sqrt{2}, -i\sqrt{2}\}$$

3. Given, $-12 < 3x - 5 \leq -4$

$$\Rightarrow -12 + 5 < 3x - 5 + 5 \leq -4 + 5 \qquad \text{(adding 5)}$$
$$\Rightarrow -7 < 3x \leq 1$$
$$\Rightarrow -\frac{7}{3} < x \leq \frac{1}{3} \qquad \text{(Dividing by 3)}$$
$$\therefore \text{ Solution set } = \left(-\frac{7}{3}, \frac{1}{3}\right]$$

4. We have, $z_1 = 1 - i$ and $z_2 = -2 + 4i$

$$\text{Now, } \frac{z_1 z_2}{\bar{z}_1} = \frac{(1-i)(-2+4i)}{1+i} = \frac{-2+2i+4i+4}{1+i}$$
$$= \frac{(2+6i)(1-i)}{1-i^2} = \frac{2+6i-2i+6}{2} = \frac{8+4i}{2} = 4+2i$$
$$\therefore \text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = 2$$

5. Let $z = \sqrt{37} + \sqrt{-19} = \sqrt{37} + \sqrt{19 \times (-1)}$

$$= \sqrt{37} + \sqrt{19}\sqrt{-1} = \sqrt{37} + i\sqrt{19}$$
$$\therefore \text{Re}(z) = \sqrt{37} \text{ and } \text{Im}(z) = \sqrt{19}$$

6. Let $z = \frac{3+2i\sin\theta}{1-2i\sin\theta}$

$$\Rightarrow z = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$
$$= \frac{3+2i\sin\theta+6i\sin\theta-4\sin^2\theta}{1+4\sin^2\theta}$$
$$= \frac{3-4\sin^2\theta}{1+4\sin^2\theta} + i\frac{8\sin\theta}{1+4\sin^2\theta}$$

For z to be purely real $\text{Im}(z) = 0$

$$\therefore \frac{8\sin\theta}{1+4\sin^2\theta} = 0 \Rightarrow \sin\theta = 0$$
$$\Rightarrow \theta = n\pi, n \in I.$$

7. Let $z=\frac{2+i}{4i+(1+i)^2}=\frac{2+i}{4i+1+i^2+2i}=\frac{2+i}{6i}$

$=\left(\frac{2+i}{6i}\right)\left(\frac{-6i}{-6i}\right)=\frac{6-12i}{36}=\frac{1}{6}-\frac{1}{3}i$

$\therefore\quad \bar{z}=\frac{1}{6}+\frac{1}{3}i$

Now, $z=\frac{1}{6}-\frac{1}{3}i$

Here $x=\frac{1}{6}, y=-\frac{1}{3}$

$\because\quad x=\frac{1}{6}>0$ and $y=-\frac{1}{3}<0$. Hence z lies in the 4^{th} quadrant.

$\therefore\quad \arg z=-\tan^{-1}\left|\frac{y}{x}\right|=-\tan^{-1}\left|\frac{-1/3}{1/6}\right|=-\tan^{-1}|-2|$

$=-\tan^{-1}2$

8. We have $x^2-14x+58=0$

Here, $a=1$, $b=-14$ and $c=58$

Now, $D=b^2-4ac=(-14)^2-4\times1\times58=196-232$

$=-36<0$

The roots are imaginary and given by

$x=\frac{-b\pm i\sqrt{4ac-b^2}}{2a}$

$\Rightarrow\quad x=\frac{14\pm i\sqrt{4\times1\times58-(-14)^2}}{2\times1}$

$\Rightarrow\quad x=\frac{14\pm i\sqrt{36}}{2}\quad\Rightarrow\quad x=\frac{14\pm6i}{2}$

$\Rightarrow\quad x=7+3i, 7-3i$

9. Let $z=x+iy$

Given $|z-2|=2|z-1|$

$\Rightarrow |x+iy-2|=2|x+iy-1|$

$\Rightarrow\quad |(x-2)+iy|=2|(x-1)+iy|$

$\Rightarrow\quad \sqrt{(x-2)^2+y^2}=2\sqrt{(x-1)^2+y^2}$

$\Rightarrow\quad (x-2)^2+y^2=4[(x-1)^2+y^2]$

$\Rightarrow\quad x^2-4x+4+y^2=4(x^2-2x+1+y^2)$

$\Rightarrow\quad x^2+y^2-4x+4=4x^2-8x+4+4y^2$

$\Rightarrow\quad 3(x^2+y^2)=4x$

$\Rightarrow\quad x^2+y^2=\frac{4}{3}x\Rightarrow|z|^2=\frac{4}{3}\mathrm{Re}(z)\qquad(\because x=\mathrm{Re}(z))$

10. Let $z=\sqrt{2}-\sqrt{2}i$

Here $x=\sqrt{2}, y=-\sqrt{2}$

$\bar{z}=\overline{\sqrt{2}-\sqrt{2}i}=\sqrt{2}+\sqrt{2}i$

and $r=|z|=|\sqrt{2}-\sqrt{2}i|=\sqrt{(\sqrt{2})^2+(\sqrt{2})^2}=\sqrt{2+2}=2$

Now, $\cos\theta=\frac{x}{r}=\frac{1}{\sqrt{2}},\quad \sin\theta=\frac{y}{r}=\frac{-1}{\sqrt{2}}$

Since $\sin\theta$ is –ve and $\cos\theta$ is +ve. Hence, θ lies in the 4^{th} quadrant.

$\Rightarrow\quad \theta=-\frac{\pi}{4}$

11. $\sqrt{4+3\sqrt{-20}}=\sqrt{4+6\sqrt{5}i}$ and

$\sqrt{4-3\sqrt{-20}}=\sqrt{4-6\sqrt{5}i}$

Let $\sqrt{4+6\sqrt{5}i}=x+iy$ ...(i)

On squaring both sides of (i), we get

$4+6\sqrt{5}i=(x+iy)^2$

$\Rightarrow\quad (4+6\sqrt{5}i)=(x^2-y^2)+i(2xy)$ ...(ii)

On comparing real parts and imaginary parts on both sides of (ii), we get

$x^2-y^2=4$ and $2xy=6\sqrt{5}$

$\therefore\quad x^2+y^2=\sqrt{(x^2-y^2)^2+4x^2y^2}=\sqrt{(4)^2+(6\sqrt{5})^2}$

$=\sqrt{16+180}=\sqrt{196}=14$

Thus, $x^2-y^2=4$ ....(iii)

and $x^2+y^2=14$ ...(iv)

On solving (iii) and (iv), we get

$x^2=9$ and $y^2=5$

$\therefore\quad x=\pm3$ and $y=\pm\sqrt{5}$

Since $xy>0$, so x and y are of the same sign.

$\therefore\quad x=3, y=\sqrt{5}$ or $x=-3, y=-\sqrt{5}$

$\therefore\quad \sqrt{4+3\sqrt{-20}}=\pm(3+\sqrt{5}i)$

Similarly, $\sqrt{4-3\sqrt{-20}}=\pm(3-\sqrt{5}i)$

Hence, $\sqrt{4+3\sqrt{-20}}+\sqrt{4-3\sqrt{-20}}=\pm6$

12. We have, $\frac{3}{2+\cos\theta+i\sin\theta}=a+ib$

$\Rightarrow\quad \frac{2+\cos\theta+i\sin\theta}{3}=\frac{1}{a+ib}=\frac{a-ib}{a^2+b^2}$

Equating real and imaginary parts, we get

$\frac{2+\cos\theta}{3}=\frac{a}{a^2+b^2}\Rightarrow\cos\theta=\frac{3a}{a^2+b^2}-2$ ...(i)

And, $\frac{\sin\theta}{3}=-\frac{b}{a^2+b^2}\Rightarrow\sin\theta=\frac{-3b}{a^2+b^2}$ ...(ii)

Squaring and adding (i) and (ii), we get

$1=\frac{9a^2}{(a^2+b^2)^2}+4-\frac{12a}{a^2+b^2}+\frac{9b^2}{(a^2+b^2)}$

$$\Rightarrow \quad 1=\frac{9(a^2+b^2)}{(a^2+b^2)^2}+4-\frac{12a}{a^2+b^2}$$

$$\Rightarrow \quad 1=\frac{9}{a^2+b^2}+4-\frac{12a}{a^2+b^2}$$

$\Rightarrow$ $a^2 + b^2 = 9 + 4(a^2 + b^2) - 12a$

$\Rightarrow$ $3(a^2 + b^2) = 12a - 9$

$\Rightarrow$ $a^2 + b^2 = 4a - 3$

13. We have, $\frac{5}{x-2}>3 \Rightarrow \frac{5}{x-2}-3>0$

$$\Rightarrow \quad \frac{5-3x+6}{x-2}>0 \quad \Rightarrow \quad \frac{11-3x}{x-2}>0.$$

$\therefore$ Either $(11 - 3x > 0$ and $x - 2 > 0)$

or $(11 - 3x < 0$ and $x - 2 < 0)$

Case I : When $11 - 3x > 0$ and $x - 2 > 0$

$\Rightarrow$ $-3x > -11$ and $x > 2$

$\Rightarrow$ $x<\frac{11}{3}$ and $x>2$

$$\Rightarrow \quad 2<x<\frac{11}{3} \qquad \text{...(i)}$$

Case II : When $11 - 3x < 0$ and $x - 2 < 0$

Now, $11 - 3x < 0$ and $x < 2$

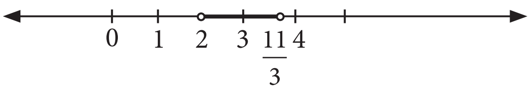
$\Rightarrow$ $-3x < -11$ and $x < 2$

$\Rightarrow$ $x>\frac{11}{3}$ and $x<2$

This is not possible, as we cannot find a real number which is greater than $\frac{11}{3}$ and less than 2.

$\therefore$ Solution set $=\{x\in R: 2<x<\frac{11}{3}\}=\left(2,\frac{11}{3}\right)$.

We can represent this set on the number line, as given below,



14. Here $(x^2+a^2)(x^2+b^2)(x^2+c^2)=|(x+ia)(x+ib)(x+ic)|^2$

Now, $(x + ia)(x + ib)(x + ic)$

$= \{x^2 - ab + i(a + b)x\}(x + ic)$

$= x^3 - abx - (a + b)cx + i\{x^2(a + b) + c(x^2 - ab)\}$

$= x^3 - (ab + bc + ca)x + i[(a + b + c)x^2 - abc]$

$$\therefore \ |(x+ia)(x+ib)(x+ic)|=|\{x^2-(ab+bc+ca)x\} +i\{(a+b+c)x^2-abc\}|$$

$$\Rightarrow \quad \sqrt{x^2+a^2}\sqrt{x^2+b^2}\sqrt{x^2+c^2} =\sqrt{\{x^2-(ab+bc+ca)x\}^2+\{(a+b+c)x^2-abc\}^2}$$

$$\Rightarrow \quad (x^2+a^2)(x^2+b^2)(x^2+c^2) =[x^2-(ab+bc+ca)x]^2+[(a+b+c)x^2-abc]^2$$

15. We have,

$x + 2y \le 10$ ...(i)

$x + y \le 6$ ...(ii)

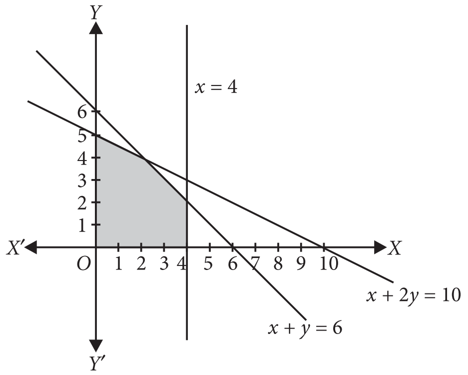
$x \le 4$ ...(iii)

$x \ge 0, y \ge 0$ ...(iv)

Now, we draw the graph of lines $x + 2y = 10$, $x + y = 6$ and $x = 4$.

The inequality (i) and (ii) represent the region below the two lines and inequality (iii) represent the region left to the line $x = 4$.

Since, $x \ge 0$, $y \ge 0$, every point of the solution lies in the first quadrant.



The shaded region in the figure represents the solution of the given inequations.

MPP-2 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.
Self check table given at the end will help you to check your readiness.

Complex Numbers and Quadratic Equations

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The Euler form of $\dfrac{2+i6\sqrt{3}}{5+i\sqrt{3}}$ is

(a) $2\cdot e^{i\frac{\pi}{6}}$ (b) $e^{i\pi/3}$

(c) $e^{-2\pi/3}$ (d) $2e^{i\frac{\pi}{3}}$

2. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

(a) 4 (b) 1

(c) 3 (d) 2

3. The locus of the centre of a circle which touches the circle $|z - z_1| = a$ and $|z - z_2| = b$, externally (where $z_1, z_2 \in C$) will be

(a) an ellipse

(b) a hyperbola

(c) a circle

(d) none of these

4. If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

(a) two real roots

(b) two positive roots

(c) two negative roots

(d) one positive and one negative root

5. Let a, b, c be the sides of a triangle where $a \neq b \neq c$ and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then

(a) $\lambda > \dfrac{5}{3}$ (b) $\lambda \in \left(\dfrac{1}{3}, \dfrac{5}{3}\right)$

(c) $\lambda \in \left(\dfrac{4}{3}, \dfrac{5}{3}\right)$

(d) none of these

6. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x - 1)^3 + 8 = 0$, are

(a) $-1, -1, -1$

(b) $-1, -1 + 2\omega, -1 - 2\omega^2$

(c) $-1, 1 + 2\omega, 1 + 2\omega^2$

(d) $-1, 1 - 2\omega, 1 - 2\omega^2$

One or More Than One Options Correct Type

7. Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and Arg z + Arg $\omega = \pi$, then z equals

(a) ω (b) $-\omega$

(c) $\bar{\omega}$ (d) $-\bar{\omega}$

8. If $1, \omega, \omega^2, ..., \omega^{n-1}$ are the n, n^{th} roots of unity, then $(2 - \omega)(2 - \omega^2) .... (2 - \omega^{n-1})$ equals

(a) $2^n - 1$

(b) ${}^nC_1 + {}^nC_2 + ... + {}^nC_n$

(c) $[{}^{2n}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n]^{1/2}$

(d) $2^n + 1$

9. If z_1, z_2, z_3, z_4 are roots of the equation $a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_0, a_1, a_2, a_3 and a_4 are real, then

(a) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of the equation

(b) z_1 is equal to at least one of $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$

(c) $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of the equation

(d) none of these

10. The equation $x^{\frac{3}{4}(\log_2 x)^2+(\log_2 x)-\frac{5}{4}} = \sqrt{2}$ has
(a) at least one real solution
(b) exactly three real solutions
(c) exactly one irrational solution
(d) complex roots

11. If α, β are roots of equation $ax^2 + bx + c = 0$, $(a, b, c \neq 0)$ then
(a) $-\frac{1}{\alpha}, -\frac{1}{\beta}$ are roots of $cx^2 - bx + a = 0$
(b) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are roots of $acx^2 + (2ac - b^2)\,x + ac = 0$
(c) $-\alpha, -\beta$ are roots of $ax^2 - bx + c = 0$
(d) α^2, β^2 are roots of $ax^2 + (2ac - b^2)\,x + c^2 = 0$

12. The equation, $\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = a(a-1)$ has
(a) four real roots if $a > 2$
(b) two real roots if $1 < a < 2$
(c) no real roots if $a < -1$
(d) four real roots for all $a < -1$.

13. The equation whose roots are n^{th} power of the roots of the equation, $x^2 - 2x\cos\theta + 1 = 0$, is given by
(a) $(x + \cos n\,\theta)^2 + \sin^2 n\theta = 0$
(b) $(x - \cos n\theta)^2 + \sin^2 n\theta = 0$
(c) $x^2 + 2x\cos n\theta + 1 = 0$
(d) $x^2 - 2x\cos n\theta + 1 = 0$

Comprehension Type

The roots of the equation $x^3 = 1$ are called cube roots of unity and the complex roots are square of each other, denoted by ω, ω^2. Observe the sum and product of cube roots of unity.

14. The value of $(1 - \omega + \omega^2)^5 + (1 - \omega^2 + \omega)^5$ equals
(a) 32 (b) -32
(c) 0 (d) none of these

15. The common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{53} + z^{31} + 1 = 0$ are
(a) ω, ω^2 (b) $1, \omega, \omega^2$
(c) $-1, \omega, \omega^2$ (d) $-\omega, -\omega^2$

Matrix Match Type

16. Let α, β, γ be the roots of $x^3 - 3ax^2 + 3bx - c = 0$. Then α, β, γ are in

Column I	Column II
(P) A.P., if	(1) $\beta = c^{1/3}$
(Q) G.P., if	(2) $\beta = \frac{c}{b}$
(R) H.P. , if	(3) $\beta = a$

	P	Q	R
(a)	2	3	1
(b)	3	1	2
(c)	1	2	3
(d)	1	3	2

Integer Answer Type

17. If α, β are the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$. If λ_1 and λ_2 are two values of λ for which the roots α, β are related by : $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, then the middle digit of $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}$ is

18. If α, β, γ are the angles such that $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$ and $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ and $z = \cos\gamma + i\sin\gamma$, then positive integer value of $xyz =$

19. If ω $(\neq 1)$ is a cube root of unity, and $P = (1+\omega)(1+\omega^2)(1+\omega^3)(1+\omega^4)(1+\omega^5) \ldots (1+\omega^{30})$. Find the sum of digits of P.

20. z_1 and z_2 are two complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular, while z_2 is not unimodular, then $|z_1|$ must be equal to ◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Class XII

FUNCTIONS

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

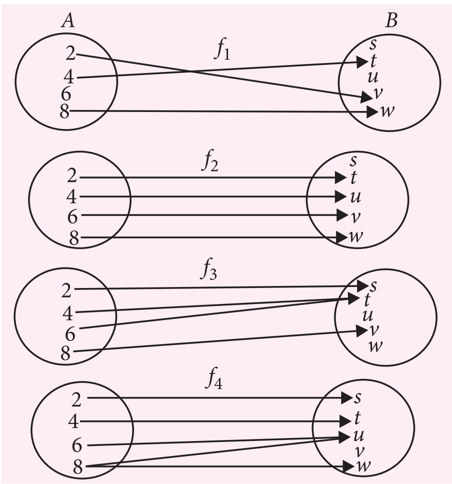
FUNCTION

Let A and B be two non–empty sets. Then a function 'f' from set A to set B is a rule which associates elements of set A to elements of set B such that an element of set A is associated to a unique element in set B.

All elements of set A are associated to elements in set B.

Terms such as "map" (or mapping), "correspondence" are used as synonyms for function. If f is a function from a set A to set B, then we write $f : A \to B$ or $A \xrightarrow{f} B$, which is read as f is a function from A to B or f maps A to B.

Example : Let $A = \{2, 4, 6, 8\}$ and $B = \{s, t, u, v, w\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements of A to elements of B as shown in the following figures.



Now see that f_1 is not a function from set A to set B, since there is an element $6 \in A$ which is not associated to any element of B, but f_2 and f_3 are the functions from A to B, because under f_2 and f_3 each element in A is associated to a unique element in B. Also, f_4 is not a function from A to B because an element $8 \in A$ is associated to two elements u and w in B.

- Let $f : A \to B$ be a function. Then,

(i) Domain : Set A is called domain of f *i.e.* set of those elements from which functions is to be defined.

(ii) Co–domain : Set B is called co–domain of f.

(iii) Range : Set of images of each element in A, is called range of f.

Note : Range $\subseteq$ Co-domain

Real Valued Function : All those functions of which domain and co-domain are subsets of R are called real valued functions. In this case for a given function we have to find domain and range.

Bounded Function : A function 'f' is said to be bounded if $|f(x)| \le m$, for some finite 'm' for every x in domain of f.

Equality of two functions : Two functions f and g are said to be equal functions, if and only if

(i) domain of f = domain of g

(ii) co–domain of f = co–domain of g

(iii) $f(x) = g(x)\ \forall\ x \in$ their domain

Example : If $f : A \to B, A = \{1, 2\}, B = \{10, 13\}, f(x) = x^2 + 9$ and $g : A \to B, g(x) = 3x + 7$, then $f = g$ because domain and co-domain of both f and g are same. Also $f(1) = 10 = g(1); f(2) = 13 = g(2)$

ALGEBRA OF FUNCTIONS

If $f : D_1 \to R$ and $g : D_2 \to R$ then, we describe functions $f + g, f - g, fg$ and f/g as follows :

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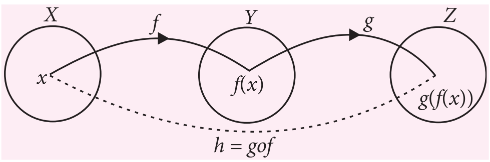
He trains IIT and Olympiad aspirants.

- $f+g : D \to R$ is a function defined by
  $(f+g)(x) = f(x) + g(x)$
- $f-g : D \to R$ is a function defined by
  $(f-g)(x) = f(x) - g(x)$
- $fg : D \to R$ is a function defined by
  $(fg)(x) = f(x)\,g(x)$
- $\frac{f}{g} : C \to R$ is a function defined by
  $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0,$$

where $D = D_1 \cap D_2$ and $C = \{x \in D : g(x) \neq 0\}$

COMPOSITE FUNCTION

Consider two functions $f : X \to Y$ and $g : Y \to Z$.
One can define $h : X \to Z$ such that
$h(x) = g\{f(x)\}$
Domain of $gof\,(x)$ *i.e.* $g\{f(x)\}$
$= \{x : x \in \text{Dom } f, f(x) \in \text{Dom } g\}$
Domain of $fog\,(x)$ *i.e* $f\{g(x)\}$
$= \{x : x \in \text{Dom } g, g(x) \in \text{Dom } f\}$



EVEN AND ODD FUNCTIONS :

Let $f : X \to Y$ be a real valued function such that for all $x \in D \Rightarrow -x \in D$ (where D = domain of f) and if $f(-x) = f(x)$ for every $x \in D$, then f is said to be an even function and if $f(-x) = -f(x)$, then f is said to be an odd function. Even functions are symmetric about the y–axis (*i.e.* if (x, y) lies on the curve, then $(-x, y)$ also lies on the curve) and odd functions are symmetric about the origin (*i.e.* if (x, y) lies on the curve, then $(-x, -y)$ also lies on the curve).

Remarks

Every function defined in symmetric interval D (*i.e.* $x \in D \Rightarrow -x \in D$) can be expressed as a sum of an even and an odd function.

$$f(x) = \left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right)$$

Let $h(x) = \left(\frac{f(x)+f(-x)}{2}\right)$ and $g(x) = \left(\frac{f(x)-f(-x)}{2}\right)$

It can now easily be shown that $h(x)$ is even and $g(x)$ is odd.

The first derivative of an even function is an odd function and vice – versa.

If $x = 0$, $x \in$ Domain of f, then for odd function $f(x)$, which is continuous at $x = 0$, $f(0) = 0$, *i.e.* if for a function $f(0) \neq 0$, then that function cannot be odd. It follows that for a differentiable even function $f'(0) = 0$ *i.e.* if for a differentiable function $f'(0) \neq 0$ then the function cannot be even.

PERIODIC FUNCTION

A function $f(x)$ is said to be periodic function if, there exists a fixed positive real number T independent of x, such that $f(x + T) = f(x)$ $\forall$ $x \in$ domain and $x + T \in$ domain.
T is called one of the period of the function.
In other words , a function is said to be periodic function if its each value is repeated after a definite interval.
Here the least positive value of T (independent of x) is called the fundamental period of the function.
Clearly, $f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots$

Note :

- $\sin x$, $\cos x$, $\sec x$ and $\text{cosec}\, x$ are periodic functions with period 2π.
- $\tan x$ and $\cot x$ are periodic functions with period π.
- $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, $|\text{cosec}\, x|$ are periodic functions with period π.
- $\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$ are periodic functions with period 2π or π according as n is odd or even.
- $\tan^n x$ and $\cot^n x$ are periodic functions with period π whether n is odd or even.

Properties of Periodic Function

If $f(x)$ is periodic with period T, then
- $f(x + c)$ is periodic with period T.
- $f(x) \pm c$ is periodic with period T.
- $f(ax + b)$ has period $\frac{T}{|a|}$, *i.e.*, period is affected only by coefficient of x ; where a, b, c are constants with a, $b \neq 0$.

If $f(x)$ and $g(x)$ be two periodic functions with period p & q respectively, then their any combination will be periodic function with one period equal to L.C.M of p & q provided L.C.M of p & q exists.

Note : All periodic functions can be analyzed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

CLASSIFICATION OF FUNCTIONS

(i) One–One Function (Injective) :

If each element in the domain of a function has a distinct image in the co–domain, then the function is said to be one–one function and is also known as injective function.
or, $f : A \to B$ is one – one
$\Leftrightarrow$ $a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$
$\Leftrightarrow$ $f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$

(ii) Many–One Function :

If there are two or more than two elements of domain having the same image then $f(x)$ is called many – one function.

If the graph of $y = f(x)$ is given and a line parallel to x–axis cuts the curve at more than one point then the function is many–one.

or, $f : A \to B$ is a many – one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

(iii) Onto Function (Surjective) :

Let $f : X \to Y$ be a function. If each element in the co–domain Y has at least one pre–image in the domain X *i.e.* Range f = Co-domain, then f is called onto.

Onto function is also called surjective and if function be both one–one and onto then function is called Bijective.

or, $f : A \to B$ is a surjection iff for each $b \in B \;\exists\; a \in A$ such that $f(a) = b$.

(iv) Into Function :

If there exist one or more than one element in the co–domain Y which is not an image of any element in the domain X. Then f is into.

In other words $f : A \to B$ is an into function if it is not an onto function.

Note :

- If domain of $f(x)$ is continuous and $\frac{dy}{dx} > 0, \forall\, x$ in domain then f is one–one, where equality exist at discrete point.
- If domain of $f(x)$ is continuous and $\frac{dy}{dx} < 0, \forall\, x$ in domain then f is one–one, where equality exist at discrete point.
- If a continuous function $f(x)$ which has either local minima or local maxima or both then $f(x)$ will be many–one.
- Every even function is many–one.
- Every periodic function is many–one.

INVERSE FUNCTION

If $f : X \to Y$ be a function defined by $y = f(x)$ such that f is both one–one and onto, then there exists a unique function $g : Y \to X$ such that for each $y \in Y$, $g(y) = x$. The function g so defined is called the inverse of f and denoted by f^{-1}. Also, f is the inverse of g and the two functions f and g are said to be inverse of each other.

$f(f^{-1}(x)) = x, \forall\, x \in Y$ and $f^{-1}(f(x)) = x, \forall\, x \in X$

Note that f and f^{-1} are symmetric about the line $y = x$.

METHOD OF FINDING INVERSE OF A FUNCTION

1. If you are asked to check whether the given function $y = f(x)$ is invertible, you need to check that $y = f(x)$ is one–one and onto.
2. If you are asked to find the inverse of a bijective function $f(x)$, you do the following : if f^{-1} be the inverse of 'f', then $f(f^{-1}(x)) = x$. Apply the formula of f on $f^{-1}(x)$ and use of the above identity to solve for $f^{-1}(x)$.

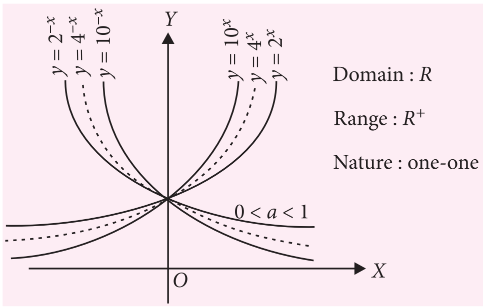
Some standard functions given below along with their inverse functions

	Function	Inverse Function
(i)	$f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$	$f^{-1} : [0, \infty) \to [0, \infty)$ defined by $f^{-1}(x) = \sqrt{x}$
(ii)	$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by $f(x) = \sin x$	$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ defined by $f^{-1}(x) = \sin^{-1} x$
(iii)	$f : [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos x$	$f^{-1} : [-1, 1] \to [0, \pi]$ defined by $f^{-1}(x) = \cos^{-1} x$

SOME ELEMENTARY FUNCTIONS

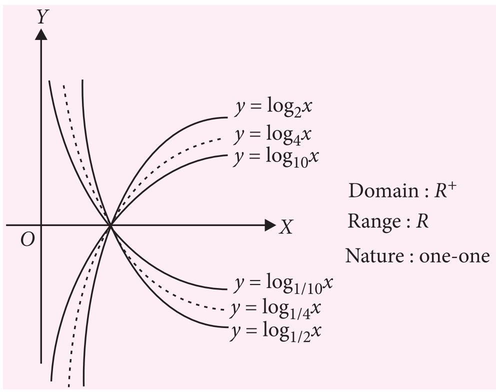
- **General Exponential Function**

If $a > 0$, $a \neq 1$ then the function defined by $f(x) = a^x$, $x \in R$ is called an Exponential Function with base a.



- **Logarithmic Function**

If $a > 0$, $a \neq 1$, then the function $y = \log_a x$, $x \in R^+$(set of positive real numbers) is called the Logarithmic Function with base a.



- **Polynomial Function**

If a function is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots + a_{n-1} x + a_n$, where n is a non negative integer and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note :

(A) A polynomial of degree one with no constant term is called an odd linear function. *i.e.* $f(x) = ax$, $a \neq 0$.

(B) There are two polynomial functions satisfying the relation $f(x)\cdot f(1/x) = f(x) + f(1/x)$.

They are

(i) $f(x) = x^n + 1$

(ii) $f(x) = 1 - x^n$, where n is a positive integer.

(C) $f(x) = c$ and $c \neq 0$ is a polynomial of degree zero.

(D) $f(x) = 0$ is a polynomial but degree not defined.

Note : Functions given in (C) and (D) are also called constant functions.

- **Algebraic Function**

y is an algebraic function of x, if it satisfies an algebraic equation of the form,

$P_0(x)y^n + P_1(x)y^{n-1} + \ldots.. + P_{n-1}(x)y + P_n(x) = 0$,

where n is a positive integer and $P_0(x)$, $P_1(x)$, ..., $P_n(x)$ are polynomials in x.

Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called a Transcendental Function.

- **Rational Function**

The function which can be written as the quotient of two polynomial functions is said to be a rational function.

If $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

and $Q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$

be two polynomial functions then a function f defined by $f(x) = \dfrac{P(x)}{Q(x)}$; $Q(x) \neq 0$ is a rational function of x.

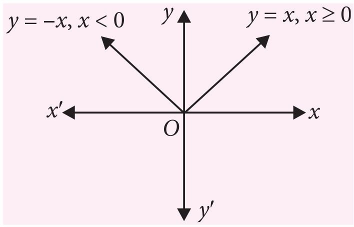
- **Identity Function**

A map $f : R \to R$ is said to be an identity function, iff $f(x) = x$, $\forall\ x \in R$.

Domain : R, Range : R

- **Modulus Function**

$$f(x) = |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$



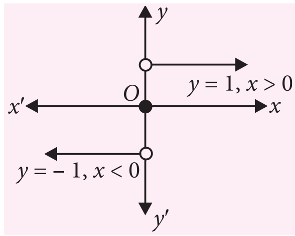
Domain : R, Range : $[0, \infty)$

It is an even continuous and many – one function.

Graph is symmetrical with respect to y–axis.

- **Signum Function**

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$



Domain : R, Range : $\{-1, 0, 1\}$. It is a many – one and discontinuous function.

- **Greatest Integer Function**

If $f(x) = k\ \forall\ x \in [k, k + 1)$, where k is any integer, then f is called greatest integer function usually denoted by $f(x) = [x]$

Properties of Greatest Integer Function

1. $x - 1 < [x] \leq x$

2. $[x] + [-x] = \begin{cases} 0; x \in I \\ -1; x \notin I \end{cases}$

3. $[x] = n \Rightarrow n \leq x < n + 1$
   $[x] \geq n \Rightarrow x \geq n,\ n \in I$
   $[x] \leq n \Rightarrow x < (n + 1),\ n \in I$

4. $[x + n] = [x] + n$, where $n \in I$

- **Fractional Part of x**

$f(x) = x - [x], x \in R$

$$i.e., f(x) = \{x\} = \begin{cases} x+1, & x \in [-1, 0) \\ x, & x \in [0, 1) \\ x-1, & x \in [1, 2) \end{cases}$$

Domain : R, Range : $[0, 1)$, Nature : Many–one

This is a periodic function with period 1. It is discontinuous at all integers.

Properties of Fractional Part of x

1. $x = [x] + \{x\}$, where [.] and {.} denotes the integral and fractional part of x respectively.

2. $\{x\} + \{-x\} = \begin{cases} 0\,; x \in I \\ 1\,; x \notin I \end{cases}$

PROBLEMS

Single Correct Answer Type

1. Range of the function $f(x)=x^2+\dfrac{1}{x^2+1}$, is

(a) $[1,\infty)$ (b) $[2,\infty)$ (c) $\left[\dfrac{3}{2},\infty\right)$ (d) $(-\infty,\infty)$

2. The range of the function

$f(x)=\log_{10}\left\{\sin^{-1}(\sqrt{x-5})+\dfrac{3\pi}{2}\right\}$ is

(a) $\left[\log\dfrac{3\pi}{2},\log 2\pi\right]$ (b) $\left[\log\dfrac{3\pi}{2},\log 3\pi\right]$

(c) $\left[\log\dfrac{3\pi}{2},\log\pi\right]$ (d) $\left[\log\dfrac{3\pi}{4},\log 2\pi\right]$

3. The range of the function $f(x)=\cos^{-1}\sqrt{\log_{[x]}\dfrac{|x|}{x}}$, where [.] = G.I.F is

(a) $\left\{\dfrac{\pi}{2}\right\}$ (b) $\{0\}$ (c) $\{\pi\}$ (d) $\{2\pi\}$

4. If $f(x)$ is a polynomial function such that $|f(x)|\le 1\ \forall\ x\in R$ and $g(x)=\dfrac{e^{f(x)}-e^{|f(x)|}}{e^{f(x)}+e^{|f(x)|}}$, then the range of $g(x)$ is

(a) $[0,1]$ (b) $\left[0,\dfrac{e^2+1}{e^2-1}\right]$

(c) $\left[0,\dfrac{e^2-1}{e^2+1}\right]$ (d) $\left[\dfrac{1-e^2}{1+e^2},0\right]$

5. The range of the function $y=[x^2]-[x]^2$, $x\in[0,2]$ where [.] denotes the integral part, is

(a) $\{0\}$ (b) $\{0,1\}$ (c) $\{1,2\}$ (d) $\{0,1,2\}$

6. If $f(x)=\dfrac{x^3+x-2}{x^3-1}$ and $g(x)=\dfrac{x^2+x+2}{x^2+x+1}$ then (D_f represents domain and R_f represents range of $f(x)$)

(a) $D_f=R-\{1\},\ R_f=\left(1,\dfrac{7}{3}\right]-\left\{\dfrac{4}{3}\right\}$

$D_g=R\quad ,R_g=\left(1,\dfrac{7}{3}\right]$

(b) $D_f=R-\{1\},\ R_f=\left(1,\dfrac{7}{3}\right]$

$D_g=R\quad ,R_g=\left[1,\dfrac{7}{3}\right]$

(c) $D_f=R-\{1\},\ R_f=\left[1,\dfrac{7}{3}\right]$

$D_g=R\quad ,R_g=\left[1,\dfrac{7}{3}\right]$

(d) $D_f=R-\{1\},\ R_f=\left(1,\dfrac{7}{3}\right]$

$D_g=R\quad ,R_g=\left(1,\dfrac{7}{3}\right]$

7. Range of $f(x)=\dfrac{\tan(\pi[x^2-x])}{1+\sin(\cos x)}$ is (where $[\cdot]$ denotes the greatest integer function

(a) $(-\infty,\infty)\sim[0,\tan 1]$ (b) $(-\infty,\infty)\sim[\tan 2,0)$

(c) $[\tan 2,\tan 1]$ (d) None of these

8. Given that $f:R^+\to R, f(x)=|x-1|$ and $g:[-1,\infty)\to R$, $g(x)=e^x$. If the function $f(g(x))$ is defined, then its domain and range respectively are

(a) $(0,\infty)$ and $[0,\infty)$ (b) $[-1,\infty)$ and $[0,\infty)$

(c) $[-1,\infty)$ and $\left[1-\dfrac{1}{e},\infty\right)$

(d) $[-1,\infty)$ and $\left[\dfrac{1}{e}-1,\infty\right)$

9. If $F(x)$ and $G(x)$ are even and odd extensions of the functions $f(x)=x|x|+\sin|x|+xe^x$, where $x\in(0,1)$, $g(x)=\cos|x|+x^2-x$, where $x\in(0,1)$ respectively to the interval $(-1,0)$ then $F(x)+G(x)$ in $(-1,0)$ is

(a) $\sin x+\cos x+xe^{-x}$

(b) $-(\sin x+\cos x+xe^{-x})$

(c) $-(\sin x+\cos x+x+xe^{-x})$

(d) $-(\sin x+\cos x+x^2+xe^{-x})$

10. Let $f(x)=\dfrac{\sin^{101}x}{\left[\dfrac{x}{\pi}\right]+\dfrac{1}{2}}$, where $[x]$ denotes the integral part of x is

(a) an odd function

(b) an even function

(c) neither odd nor even function

(d) both odd and even function

11. Let Z denote the set of all integers. Define $f:Z\to Z$ by $f(x)=\begin{cases}x/2, & (x \text{ is even})\\ 0, & (x \text{ is odd})\end{cases}$ then f is

(a) onto but not one-one

(b) one-one but not onto

(c) one-one and onto

(d) neither one-one nor onto

12. Let a function $f : R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then f is
(a) one-to-one and onto
(b) one-to-one not onto
(c) onto but not one-to-one
(d) neither one-to-one or onto

13. $f(x) = x^2 - x + 1, x \geq \frac{1}{2}$ and $g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$ are mutually inverse then the number of solutions of the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{\left(x - \frac{3}{4}\right)}$ is
(a) 0 (b) 2 (c) 1 (d) ∞

14. The range of the function
$f(x) = \sqrt{\tan^{-1}\sqrt{x^2 - 1} - \sec^{-1} x}$ is
(a) $[0, \sqrt{\pi}]$ (b) $\{0\}$
(c) $\left[0, \sqrt{\frac{\pi}{2}}\right]$ (d) $\left[0, \sqrt{\frac{\pi}{4}}\right]$

15. Let $f(x) = \sqrt{|x| - \{x\}}$ (where $\{\cdot\}$ denotes the fractional part of x) and X, Y be its domain and range respectively. Then
(a) $X = \left(-\infty, \frac{1}{2}\right]$ and $Y = \left[\frac{1}{2}, \infty\right)$
(b) $X = \left(-\infty, -\frac{1}{2}\right]$ and $Y = \left[\frac{1}{2}, \infty\right)$
(c) $X = \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y = [0, \infty)$
(d) $X = \left(-\infty, -\frac{1}{2}\right]$ and $Y = [0, \infty)$

Multiple Correct Answer Type

16. If $f(x) = \cos[\pi^2]\, x + \cos[-\pi^2]\, x$, where [.] is greatest integer function, then
(a) $f\left(\frac{\pi}{2}\right) = -1$ (b) $f(\pi) = 1$
(c) $f(-\pi) = 0$ (d) $f\left(\frac{\pi}{4}\right) = 1$

17. Which of the following functions $f : R \to R$ are bijective ?
(a) $f(x) = x \sin x$ (b) $f(x) = x - \sin^2 x$
(c) $f(x) = x + \sqrt{x^2}$ (d) $f(x) = -x + \cos^2 x$

18. A function $f : R \to R$ is defined by
$f(x + y) - kxy = f(x) + 2y^2 \; \forall \; x, y \in R$
And $f(1) = 2$; $f(2) = 8$, where k is some constant, then $f(x + y) \cdot f\left(\frac{1}{x + y}\right)$ is equal to (where $x + y \neq 0$)
(a) 1 (b) 4 (c) k (d) $f(1)$

19. Let $f : N \to N$ satisfying the conditions
(i) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$
(ii) $1900 < f(1990) < 2000$, where $[x]$ denotes the integral part of x. Then which of the following are true?
(a) $f(1990) = 1990$ (b) $f(1990) = 1994$
(c) $f(1990) = 2004$ (d) $f(1990) = 1904$

20. Let $y = f(x) = \frac{x + 2}{x - 1}$ then
(a) $x = f(y)$ (b) $f(1) = 3$
(c) $f(2) = 5$ (d) f is rational function

21. Let $f : \{x, y, z\} \to \{1, 2, 3\}$ is a one-one function. It is given that exactly one of the statements is true and the other two are false. $S_1 : f(x) = 1$, $S_2 : f(y) \neq 1$, $S_3 : f(z) \neq 2$. Then
(a) $f(x) = 2$ (b) $f(y) = 1$
(c) $f(z) = 3$ (d) $f^{-1}(1) = y$

22. Out of all the possible functions f satisfying $f(x) + f(x + 4) = f(x + 2) + f(x + 6)$ which cannot be the fundamental period of $f(x)$?
(a) 1 (b) 3 (c) 8 (d) 16

23. If $f(x) = \cos\left[\frac{\pi^2}{2}\right]x + \sin\left[-\frac{\pi^2}{2}\right]x$, $[\cdot]$ denoting the greatest integer function then
(a) $f(0) = 1$ (b) $f\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3} + 1}$
(c) $f(\pi/2) = 0$ (d) $f(\pi) = 0$

24. If the following functions are defined from $[-1, 1]$ to $[-1, 1]$, identify which are bijective?
(a) $\sin(\sin^{-1} x)$ (b) $\frac{2}{\pi}\sin^{-1}(\sin x)$
(c) $(\mathrm{sgn}\,(x) \ln(e^x))$ (d) $x^3\,(\mathrm{sgn}\,(x))$

25. If $2f(x) + xf\left(\frac{1}{x}\right) - 2f\left(\left|\sqrt{2}\sin\pi\left(x + \frac{1}{4}\right)\right|\right)$
$= 4\cos^2\left(\frac{\pi x}{2}\right) + x\cos\frac{\pi}{x}, \forall x \in R - \{0\}$ then which of the following statement(s) is/are true?
(a) $f(2) + f\left(\frac{1}{2}\right) = 1$ (b) $f(2) + f(1) = 0$
(c) $f(2) + f(1) = f\left(\frac{1}{2}\right)$ (d) $f(1)\, f\left(\frac{1}{2}\right) f(2) = 1$

26. The function $f(x)=\cos^{-1}\left(\frac{2[|\sin x|+|\cos x|]}{\sin^2 x+2\sin x+\frac{11}{4}}\right)$ is defined if x belongs to (where [.] represents greatest integer function)

(a) $\left[0,\frac{7\pi}{6}\right]$ (b) $\left[0,\frac{\pi}{6}\right]$

(c) $\left[\frac{11\pi}{6},2\pi\right]$ (d) $[\pi, 2\pi]$

27. If $y = \sin^2x + \cos^4x$ then for all real x

(a) the minimum value of y is 2

(b) the minimum value of y is 3/4

(c) $y \le 1$

(d) $y \ge 1/4$

Comprehension Type

Paragraph for Question No. 28 to 30

Let $f(x) = \log_{\{x\}}[x]$, $g(x) = \log_{[x]}\{x\}$, $h(x) = \log_{\{x\}}\{x\}$ (where $[\cdot]$, $\{\cdot\}$ denotes greatest integer function and fractional part).

28. For $x \in (1, 5)$, $f(x)$ is not defined at how many points?

(a) 5 (b) 4 (c) 3 (d) 2

29. If $A = \{x : x \in$ domain of $f(x)\}$ and $B = \{x : x \in$ domain of $g(x)\}$, then $A - B$ will be

(a) (2, 3) (b) (1, 3)

(c) (1, 2) (d) none of these

30. Domain of $h(x)$ is

(a) R (b) I

(c) $R - I$ (d) $R^+ - I$

Paragraph for Question No. 31 to 33

For $x \neq 0, 1$ define

$f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x) = 1/(1 - x)$, $f_5(x) = (x - 1)/x$, $f_6(x) = x/(x - 1)$

This family of functions is closed under composition that is, the composition of any two of these functions is again one of these.

31. Let F be a function such that $f_1 o F = f_4$. Then F is equal to

(a) f_1 (b) f_2 (c) f_3 (d) f_4

32. Let G be a function such that $G\ o\ f_3 = f_6$. Then G is equal to

(a) f_5 (b) f_4 (c) f_3 (d) f_2

33. Let H be a function such that $f_4\ o\ H = f_5$. Then H is equal to

(a) f_2 (b) f_4 (c) f_5 (d) f_6

Paragraph for Question No. 34 to 36

Let $f: R - \{0, 1\} \to R$ be a function satisfying the relation $f(x)+f\left(\frac{x-1}{x}\right)=x$ for all $x \in R - \{0, 1\}$. Based on this, answer the following questions.

34. $f(x)$ is equal to

(a) $\frac{1}{2}\left[x+\frac{1}{1-x}-\frac{x-1}{x}\right]$ (b) $\frac{1}{2}\left[x-\frac{1}{1-x}+\frac{x-1}{x}\right]$

(c) $\frac{1}{2}\left[x-\frac{1}{1-x}-\frac{x-1}{x}\right]$ (d) $\frac{1}{2}\left[x+\frac{1}{1-x}+\frac{x-1}{x}\right]$

35. $f(-1)$ is equal to

(a) $\frac{3}{4}$ (b) $\frac{-3}{4}$ (c) $\frac{5}{4}$ (d) $\frac{-5}{4}$

36. $f\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{5}{4}$ (b) $\frac{-7}{4}$ (c) $\frac{7}{4}$ (d) $\frac{9}{4}$

Paragraph for Question No. 37 to 39

If a function $y = f(x)$; $f : A \to B$ then the set A is called as domain of the function and B is called co-domain of the function. For all $x \in A$, the values of y thus obtained comprise the set 'C' where C is called as range of function.

37. The domain of the function $f(x)=\frac{1}{\ln[\cos^{-1}x]}$ is where $[\cdot]$ indicates greatest integer function

(a) [0, 1] (b) [–1, cos 2]

(c) [–1, cos 3) ∪ (cos 3, cos 4)

(d) [–1, cos 3) ∪ (cos 3, cos 2)

38. The domain of the function $f(x)=\sqrt{\cos(\sin x)}+\sqrt{\log_x\{x\}}$ where {.} indicates fractional part function

(a) $[1, \pi)$ (b) $(0, 2\pi) - [1, \pi)$

(c) $\left(0,\frac{\pi}{2}\right)-\{1\}$ (d) (0, 1)

39. The range of the function of $f(x)=\sin^{-1}\sqrt{x^2+x+1}$ is

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $\left[0,\frac{\pi}{3}\right]$ (c) $\left[\frac{\pi}{3},\frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$

Matrix-Match Type

40. Match Column I with Column II and select the correct answer using the code given below the list :

	Column-I		Column-II
(A)	The number of integral solutions of $\frac{x+2}{x^2+1} > \frac{1}{2}$ is	(p)	5
(B)	The number of real values of x satisfying $1 + \lvert e^x - 1\rvert = e^x(e^x - 2)$	(q)	2
(C)	The number of integral solutions of $\lvert[x] - 2x\rvert = 4$ where $[\cdot]$ denotes G.I.F is	(r)	1
(D)	The number of integral solutions satisfying $\lvert\lvert x - 1\rvert - 1\rvert \le 1$	(s)	3

41. Match the functions specified in Column – I with the subset of their domain given in Column – II

	Column-I		Column-II
(A)	$f(x) = \sin^{-1}[x-1] + \sec^{-1}[x-1]$ where $[\cdot]$ represents greatest integer function	(p)	$[0, 1)$
(B)	$f(x) = \sqrt{(x+2)(5-x)} - \frac{1}{\sqrt{x^2-4}}$	(q)	$(2, 5]$
(C)	$f(x) = \log_{10}\log_{10}(1+x^3)$	(r)	$(-\infty, \infty)$
(D)	$f(x) = \log_e(x^2 - x + 1)$	(s)	$(2, 3)$
		(t)	$(0, \infty)$

42. Let $f, g : R \to R$ be the function defined by $f(x) = x^2 + 1$ and $g(x) = 2[x] - 1$, where $[x]$ is the largest integer $\le x$. Then match the items given in Column I with those in Column II.

	Column-I		Column-II
(A)	$(gof)\left(\frac{1}{2}\right)$	(p)	3
(B)	$(fog)\left(\frac{3}{2}\right)$	(q)	0
(C)	$(fogof)\left(\frac{3}{4}\right)$	(r)	2
(D)	$(gofog)\left(\frac{2}{3}\right)$	(s)	1

Integer Answer Type

43. A function $f : R \to R$ is defined by $f(x + y) - kxy = f(x) + 2y^2 \ \forall \ x, y \in R$ and $f(1) = 2; f(2) = 8$, where 'k' is some real constant then $f(x+y)\cdot f\left(\frac{1}{x+y}\right) =$

44. If $f : R \to R$ satisfying
$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$, for all $x, y \in R$, then $\frac{-f(10)}{7} =$

45. If $f(x) = \frac{2010x+163}{165x-2010}, x > 0$ and $x \ne \frac{2010}{165}$ then the least value of $f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right)$ is

46. $f(x) = \frac{ax+b}{cx+d} \ \forall \ x \in R - \left\{\frac{-d}{c}\right\}$ and if $f(5) = 5$, $f(13) = 13$ and $f[f(x)] = x$ for all x. Then range of $f(x) = R - \{x\}$, then $x =$

47. If $f(x) = x^3 - 12x + P$ and $P \in \{1, 2, 3, 4, 5, \ldots, 15\}$ and for each 'P', the number of real roots of equation $f(x) = 0$ is denoted by Q then $\frac{1}{5}\Sigma Q$ is equal to

48. Let $f(n)$ denote the square of the sum of the digits of natural number n, where $f^2(n)$ denote $f(f(n))$, $f^3(n)$ denote $f(f(f(n)))$ and so on. The value of $\frac{f^{2011}(2011) - f^{2010}(2011)}{f^{2013}(2011) - f^{2012}(2011)} =$

49. If 'f' is a polynomial function satisfying the condition $f(\tan x) + f(\cot x) = f(\tan x)\cdot f(\cot x) \ \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ and $f(2) = 9$, then the value of $\frac{f'(2)}{6}$ is

50. If f is a polynomial function satisfying
$2 + f(x)\cdot f(y) = f(x) + f(y) + f(xy) \ \forall \ x, y \in R$
and if $f(2) = 5$, then the value of $f(f(1))$ is

SOLUTIONS

1. **(a) :** We have, $f(x) = x^2 + 1 + \frac{1}{x^2+1} - 1$

$\Rightarrow \quad x^2 + 1 + \frac{1}{x^2+1} \ge 2 \quad [\because \text{ A.M.} \ge \text{G.M.}]$

$\Rightarrow \quad x^2 + \frac{1}{x^2+1} \ge 1 \qquad \therefore \ f(x) \in [1, \infty)$

2. **(a) :** Let $f(x)=\log_{10}\left\{\sin^{-1}(\sqrt{x-5})+\frac{3\pi}{2}\right\}$

The function is defined if (i) $x-5\ge 0$

(ii) $-1\le\sqrt{x-5}\le 1$ and (iii) $\sin^{-1}(\sqrt{x-5})+\frac{3\pi}{2}>0$

Now (i) $\Rightarrow x\ge 5$

(ii) $\Rightarrow 0\le x-5\le 1 \Rightarrow 5\le x\le 6$

(iii) is satisfied by virtue of (ii).

Hence, considering (i) and (ii), we find that the domain of the function is $D_f=[5,6]$.

Let $y_1=\sin^{-1}(\sqrt{x-5})$ and $y_2=\sin^{-1}(\sqrt{x-5})+\frac{3\pi}{2}$

so that $y=\log_{10}(y_2)$ where $y_2=y_1+\frac{3\pi}{2}$

Now, for y_1, since $x\in[5,6]$, $y_1\ge 0$ so that

$$0\le y_1\le\frac{\pi}{2}\left(\because -\frac{\pi}{2}\le\sin^{-1}(x)\le\frac{\pi}{2}\right)$$

Consequently, $0+\frac{3\pi}{2}\le y_1+\frac{3\pi}{2}\le\frac{\pi}{2}+\frac{3\pi}{2}$

$$\Rightarrow \frac{3\pi}{2}\le y_2\le 2\pi$$

$\because$ log is an increasing function in given domain

$$\Rightarrow \log\left(\frac{3\pi}{2}\right)\le\log(y_2)\le\log(2\pi)$$

Hence, the range of $f(x)$ is $\left[\log\frac{3\pi}{2},\ \log 2\pi\right]$

3. **(a) :** The function is defined if

(i) $[x]>0$ and $[x]\ne 1$

(ii) $\frac{|x|}{x}>0$

(iii) $\log_{[x]}\frac{|x|}{x}\ge 0$ (iv) $0\le\sqrt{\log_{[x]}\frac{|x|}{x}}\le 1$

Now, (i) $\Rightarrow [x]=2,3,\ldots$ *i.e.* $x\ge 2$ *i.e.* the domain of the function is $[2,\infty)$.

For this value of x (≥ 2) (ii) is true,

(iii) is also true and $\sqrt{\log_{[x]}\frac{|x|}{x}}=\sqrt{\log_{[x]}1}=0$

Hence $f(x)=\cos^{-1}(0)=\frac{\pi}{2}$

Hence, the range of the function is $\left\{\frac{\pi}{2}\right\}$.

4. **(d):** For $0\le f(x)\le 1$, $g(x)=0$

For $-1\le f(x)<0$,

$$g(x)=\frac{e^{2f(x)}-1}{e^{2f(x)}+1}\Rightarrow g(x)\in\left[\frac{1-e^2}{1+e^2},0\right)$$

$\therefore$ Range of $g(x)=\left[\frac{1-e^2}{1+e^2},0\right]$

5. **(d):** We have, $y=[x^2]-[x]^2$, $x\in[0,2]$

$$i.e.,\ y=\begin{cases}[x^2], & 0\le x<1\\ [x^2]-1, & 1\le x<2\end{cases}$$

Now, $y=[x^2]-4, x=2\Rightarrow y=0, x=2$

$$\therefore\quad y=\begin{cases}0, & 0\le x<1\\ 1-1=0, & 1\le x<\sqrt{2}\\ 2-1=1, & \sqrt{2}\le x<\sqrt{3}\\ 3-1=2, & \sqrt{3}\le x<2\\ 0, & x=2\end{cases}$$

Hence, the range is $\{0, 1, 2\}$

6. **(d):** $f(x)=\frac{x^3+x-2}{x^3-1}$

$$\Rightarrow\quad f(x)=\left(\frac{x-1}{x-1}\right)\left(\frac{x^2+x+2}{x^2+x+1}\right)=\frac{x^2+x+2}{x^2+x+1}$$

when $x\ne 1$

Hence, range is $\left(1,\frac{7}{3}\right]$.

7. **(d):** $f(x)=\frac{\tan(\pi[x^2-x])}{1+\sin x(\cos x)}=\{0\}$ because $[x^2-x]$ is integer.

8. **(b):** $f(x)=|x-1|=\begin{cases}1-x, & 0<x<1\\ x-1, & x\ge 1\end{cases}$, $g(x)=e^x$, $x\ge -1$

$$fog(x)=\begin{cases}1-g(x), & 0<g(x)<1 \text{ is } -1\le x<0\\ g(x)-1, & g(x)\ge 1 \text{ is } 0\le x\end{cases}$$

$$=\begin{cases}1-e^x, & -1\le x<0\\ e^x-1, & x\ge 0\end{cases}$$

$\therefore$ Domain $=[-1,\infty)$

fog is decreasing in $[-1,0)$ and increasing in $[0,\infty)$

$x\to\infty\Rightarrow fog(x)\to\infty\Rightarrow fog(0)=0$. Range $=[0,\infty)$

9. **(c) :** $f(x)=f(-x)$ where $f(x)=x^2+\sin x+xe^x$

$F(x)=x^2-\sin x-xe^{-x}$ ...(1)

$g(x)=-g(-x)$, where $g(x)=\cos x+x^2-x$

$G(x)=-(\cos x+x^2+x)=-\cos x-x^2-x$

$\therefore$ $F(x)+G(x)=-\sin x-xe^{-x}-\cos x-x$

$=-(\sin x+\cos x+x+xe^{-x})$

10. **(b):** When $x=n\pi$, $f(x)=0$ and $f(-x)=0$

$\therefore$ $f(-x)=f(x)$

When $x\ne n\pi$, $n\in I$, $\frac{x}{\pi}\notin I$

$$\therefore\ \left[\frac{x}{\pi}\right]+\left[-\frac{x}{\pi}\right]=-1 \Rightarrow \left[-\frac{x}{\pi}\right]=-1-\left[\frac{x}{\pi}\right]$$

$$\Rightarrow\ \left[-\frac{x}{\pi}\right]+\frac{1}{2}=-\left[\frac{x}{\pi}\right]-\frac{1}{2}=-\left(\left[\frac{x}{\pi}\right]+\frac{1}{2}\right)$$

$$\text{Now } f(-x)=\frac{\sin^{101}(-x)}{\left[-\frac{x}{\pi}\right]+\frac{1}{2}}=\frac{-\sin^{101}x}{-\left(\left[\frac{x}{\pi}\right]+\frac{1}{2}\right)}=\frac{\sin^{101}x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}=f(x)$$

Hence in all cases $f(-x) = f(x)$

11. (a) : $f(0) = 0, f(\pm 1) = 0, f(\pm 2) = \pm 1, f(\pm 3) = 0,$ $f(\pm 4) = \pm 2, ...$

Range $= Z$ and $f(\pm 1) = 0 \Rightarrow f$ is onto but not one-one.

12. (a) : Since $f'(x) = 2 + \cos x$ for all $x \in R$, so f is one-to-one. Moreover, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, hence the range of f is R. Therefore, f is onto as well.

13. (c) : The function $y = f(x) = x^2 - x + 1$

$=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$ increases in the interval $\left[\frac{1}{2},\infty\right)$

and x varying in the indicated interval, we have

$$y=f(x)\geq\frac{3}{4}\ \text{i.e., } y\in\left[\frac{3}{4},\infty\right)$$

$$\therefore\ x^2-x+1-y=0 \Rightarrow x=\frac{1}{2}+\sqrt{\left(y-\frac{3}{4}\right)}=g(y)$$

$$\therefore\ y=g^{-1}(x) \Rightarrow f(x)=g^{-1}(x)$$

Since the graphs of the original and inverse functions can intersect only on the straight line $y = x$.

$$\therefore\ x=f(x) \Rightarrow x=x^2-x+1 \quad \therefore\ x=1$$

14. (b): $\tan^{-1}\sqrt{x^2-1}\geq\sec^{-1}x$ and $x^2\geq 1$

$\Rightarrow\ x\geq 1$

Let $x = \sec\theta$ then

$$\tan^{-1}\sqrt{x^2-1}=\tan^{-1}(\tan\theta)=\theta=\sec^{-1}x$$

So range of $f(x)$ is $\{0\}$.

15. (c) : $x\geq 0 \Rightarrow |x|-\{x\}=x-\{x\}\geq 0$

$x<0 \Rightarrow |x|-\{x\}=-(x+\{x\})=[x]-2x\geq 0$ only

if $2x\leq[x] \Rightarrow$ if $x\leq-\frac{1}{2}$

16. (a, c) : $9 < \pi^2 < 10 \Rightarrow [\pi^2] = 9$

$-10 < -\pi^2 < -9 \Rightarrow [-\pi^2] = -10$

$\therefore\ f(x) = \cos 9x + \cos(-10)x = \cos 9x + \cos 10x$

$$f\left(\frac{\pi}{2}\right)=\cos 5\pi=-1,\ f(\pi)=0,\ f(-\pi)=0$$

$$f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

17. (b, d) : (a) f is not one-one as f cuts x-axis twice in $(0, 2\pi]$

In fact f is continuous and achieves every real number infinite times.

(b) f is monotonic as $f'(x) = 1 - \sin 2x \geq 0\ \forall\ x \in R$

and $\lim_{x\to-\infty} f = -\infty;\ \lim_{x\to\infty} f = \infty\ \therefore\ f$ is continuous.

Hence, f is bijective.

(c) $f(x)=x+|x|=\begin{cases}2x, & \text{if } x\geq 0\\ 0, & \text{if } x\leq 0\end{cases}$ is not one-one as f is a constant function for $x \leq 0$

(d) f is monotonic as $f'(x) \leq 0\ \forall\ x \in R$ and as in (b), f is bijective.

18. (b, c) : Given, $f(x + y) - kxy = f(x) + 2y^2$

Replace y by $-x$, then $f(0) + kx^2 = f(x) + 2x^2$

$\Rightarrow\ f(x) = f(0) + kx^2 - 2x^2$ ...(1)

Now, $f(1) = f(0) + k - 2 = 2 \Rightarrow f(0) = -k + 4$

and $f(2) = f(0) + 4k - 8 = 8 \Rightarrow f(0) = -4k + 16$

Which gives $k = 4$ and $f(0) = 0$

Thus, from (1), $f(x) = 2x^2$

$$\therefore\ f(x+y)f\left(\frac{1}{x+y}\right)=4=k$$

19. (b, d) : Given, $1900 < f(1990) < 2000$

$$\Rightarrow\ \frac{1900}{90}<\frac{f(1990)}{90}<\frac{2000}{90}$$

$$\Rightarrow\ \left[\frac{1900}{90}\right]\leq\left[\frac{f(1990)}{90}\right]\leq\left[\frac{2000}{90}\right]$$

$$\Rightarrow\ 21\leq\left[\frac{f(1990)}{90}\right]\leq 22 \Rightarrow \left[\frac{f(1990)}{90}\right]=21, \text{ or } 22$$

$$\text{Given, } x-f(x)=19\left[\frac{x}{19}\right]-90\left[\frac{f(x)}{90}\right] \quad \text{... (1)}$$

$$\therefore\ 1990-f(1990)=19\left[\frac{1990}{19}\right]-90\left[\frac{f(1990)}{90}\right]$$

$$\Rightarrow\ 1990-f(1990)=\begin{cases}19\times 104-90\times 21\\ 19\times 104-90\times 22\end{cases}$$

$$\Rightarrow\ 1990-f(1990)=\begin{cases}1976-1890=86\\ 1976-1980=-4\end{cases}$$

$\Rightarrow\ f(1990) = 1904, 1994$

20. (a, d) : $y = f(x) = \frac{x+2}{x-1}$, $f(y) = \frac{y+2}{y-1} = \frac{\frac{x+2}{x-1}+2}{\frac{x+2}{x-1}-1} = x$

$f(x) = \frac{x+2}{x-1}$, a rational function since it is ratio of two polynomials.

21. (a, b, c, d) : If S_1 is true, S_2 & S_3 are false
$\Rightarrow f(x) = f(y) = 1$, which is a contradiction
If S_2 is true, S_1 & S_3 are false $\Rightarrow f(z) = 2, f(x) \neq 1; f(y) \neq 1$ which is false
If S_3 is true, S_1 & S_2 are false
$\Rightarrow f(x) \neq 1, f(y) = 1, f(z) \neq 2$
$\Rightarrow f(y) = 1, f(z) = 3, f(x) = 2$ and $f^{-1}(1) = y$

22. (b, d) : If $f(x) + f(x+4) = f(x+2) + f(x+6)$ ...(1)
$x \to x + 2$
$\Rightarrow f(x+2) + f(x+6) = f(x+4) + f(x+8)$ ...(2)
From (1) and (2), $f(x) = f(x+8)$
$\Rightarrow$ 16 cannot be fundamental period
Similarly 3 cannot be fundamental period.
As if $f(x) = f(x+3)$, from (1) it implies that $f(x) = f(x+1)$.

23. (a, b, c) : $f(x) = \cos\left[\frac{\pi^2}{2}\right]x + \sin\left[-\frac{\pi^2}{2}\right]x$

$\left[\frac{\pi^2}{2}\right] = 4$ & $\left[-\frac{\pi^2}{2}\right] = -5$

$\therefore$ $f(x) = \cos 4x - \sin 5x$

$\therefore$ $f(0) = 1$, $f\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{1}{\sqrt{3}+1}$

$f(\pi/2) = 0, f(\pi) = 1$

24. (a) : (a) $f(x) = \sin(\sin^{-1}x) = x$, $\forall x \in [-1, 1]$ which is one – one and onto

(b) $f(x) = \frac{2}{\pi}\sin^{-1}(\sin x) = \frac{2}{\pi}x$

The range of the function for $x \in [-1\ 1]$ is $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$ which is a subset of $[-1, 1]$.
Hence the function is one-one but not onto. Hence not bijective

(c) $f(x) = (\text{Sgn}(x))\ \log(e^x) = (\text{Sgn}(x))\ x = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$

This function has the range [0, 1] which is a subset of [–1, 1]. Hence the function is into. Also the function is many one.

(d) $f(x) = x^3\,\text{sgn}(x) = \begin{cases} x^3, & x > 0 \\ -x^3, & x < 0 \\ 0, & x = 0 \end{cases}$ which is many one and into

25. (a, b, c) : Replace x by 2, we get

$2f(2) + 2f\left(\frac{1}{2}\right) - 2f(1) = 4 \Rightarrow f(2) + f\left(\frac{1}{2}\right) = 2 + f(1)$ ...(1)

Replace x by 1, $f(1) = -1$ ...(2)

Replace x by $\frac{1}{2}$, $2f\left(\frac{1}{2}\right) + \frac{1}{2}f(2) + 2 = \frac{5}{2}$

Solving (1) & (3) we get $f\left(\frac{1}{2}\right) = 0; f(2) = 1$

26. (a, b, c) : $1 \le |\sin x| + |\cos x| \le \sqrt{2}$
$[|\sin x| + |\cos x|] = 1$
$f(x)$ is defined if $\sin^2 x + 2\sin x + \frac{11}{4} \ge 2$

$\Rightarrow$ $(\sin x + 1)^2 \ge \frac{1}{4} \Rightarrow \sin x + 1 \ge \frac{1}{2}$ or $\sin x + 1 \le -\frac{1}{2}$

$\Rightarrow$ $\sin x \ge -\frac{1}{2}$ or $\sin x \le -\frac{3}{2}$ (which is not true)

27. (b, c) : $y = \sin^2 x + \cos^2 x\,(1 - \sin^2 x)$
$= 1 - \sin^2 x \cos^2 x = 1 - \frac{1}{4}\sin^2 2x$

$\therefore$ Minimum value of $y = 1 - \frac{1}{4} = \frac{3}{4}$

Maximum value of $y = 1 - 0 = 1$

28. (c) : $\{x\} \neq 0 \Rightarrow x \neq 2, 3, 4$

29. (c) : $A = \{x : x \notin I^+\}$
$B = \{x : x \notin I^+ \cap (2, \infty)\}$
$A - B = (1, 2)$

30. (c) : $\{x\} \neq 0 \Rightarrow x \in R - I$

(31-33) : We know that
$f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_5$.

31. (d): $f_1 o F = f_4 \Rightarrow F = f_1^{-1} o f_4 = f_1 o f_4$.

Thus $F(x) = f_1(1/(1-x)) = 1/(1-x) = f_4(x)$

32. (a): $G o f_3 = f_6 \Rightarrow G = f_6 o f_3^{-1} = f_6 o f_3$

So $G(x) = f_6(1-x) = (1-x)/((1-x)-1) = (x-1)/x = f_5(x)$

33. (b): $f_4 o H = f_5 \Rightarrow H = f_4^{-1} o f_5 = f_5 o f_5$

Therefore, $H(x) = f_5(f_5(x))$

$= f_5((x-1)/x) = \frac{((x-1)/x) - 1}{(x-1)/x} = \frac{1}{1-x} = f_4(x)$

34. (a) **35. (d)** **36. (c)**

(34-36) : Given that, $f(x) + f\left(\frac{x-1}{x}\right) = x$ ...(1)

For all $x \neq 0, 1$ replacing x with $\frac{x-1}{x}$ on both sides, we get

$$f\left(\frac{x-1}{x}\right)+f\left(\frac{[(x-1)/x]-1}{(x-1)/x}\right)=\frac{x-1}{x}$$

$$\Rightarrow \quad f\left(\frac{x-1}{x}\right)+f\left(\frac{1}{1-x}\right)=\frac{x-1}{x} \qquad \text{...(2)}$$

Again replacing x with $\frac{x-1}{x}$, we get

$$f\left(\frac{1}{1-x}\right)+f(x)=\frac{1}{1-x} \qquad \text{...(3)}$$

Then by taking eq. (1) + eq. (3) – eq. (2), we get

$$2f(x)=x+\frac{1}{1-x}-\frac{x-1}{x}$$

or $\quad f(x)=\frac{1}{2}\left[x+\frac{1}{1-x}-\frac{x-1}{x}\right] \qquad \text{...(4)}$

Substituting $x = -1$ and $x = \frac{1}{2}$ in eq. (4) we get

$$f(-1)=\frac{-5}{4} \text{ and } f\left(\frac{1}{2}\right)=\frac{7}{4}$$

37. (b) : Case I : $[\cos^{-1} x] \neq 1 \Rightarrow \cos^{-1}x < 0$ or $\cos^{-1}x \geq 2$

$\Rightarrow \quad x > 1$ or $x \leq \cos 2$

Case II : $[\cos^{-1}x] > 0$

$-1 \leq x \leq 1, \;\cos^{-1}x \geq 1$

$x < \cos 1 \;\Rightarrow\;$ Domain of $f(x)=[-1, \cos 2]$

38. (d) : cos (sin$x$) is defined for all real x.

$\log_x \{x\} \geq 0$

Case I : $0 < x < 1$

$\{x\} \leq 1; x \in (0, 1)$

Case II : $x > 1$

$\{x\} \geq 1$ but $0 < \{x\} < 1$ is not possible.

39. (c) : Least value of $x^2 + x + 1 = 3/4$ (at $x = -1/2$)

$\therefore\;$ least value of $\sqrt{x^2+x+1}=\frac{\sqrt{3}}{2}$

Also $\sqrt{x^2+x+1} \leq 1$

Thus $\frac{\sqrt{3}}{2} \leq \sqrt{x^2+x+1} \leq 1$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \leq \sin^{-1}\sqrt{x^2+x+1} \;\leq \sin^{-1}(1)$$

$$\frac{\pi}{3} \leq \sin\sqrt{x^2+x+1} \leq \frac{\pi}{2} \quad \text{Range of } f=\left[\frac{\pi}{3},\frac{\pi}{2}\right]$$

40. (A) → (s), (B) → (r), (C) → (q), (D) → (p)

(A) $\frac{x+2}{x^2+1} > \frac{1}{2} \Rightarrow 2x + 4 > x^2 + 1$

$\Rightarrow x^2 - 2x - 3 < 0 \Rightarrow -1 < x < 3 \Rightarrow x = 0, 1, 2$

(B) If $e^x > 1 \;\Rightarrow\; 1 + e^x - 1 = e^{2x} - 2e^x$

$\Rightarrow\; e^{2x} - 3e^x = 0 \;\Rightarrow\; e^x = 3 \Rightarrow x = \log_e 3$

If $e^x < 1 \;\Rightarrow\; 1 - e^x + 1 = e^{2x} - 2e^x$

$\Rightarrow\; e^{2x} - e^x - 2 = 0 \Rightarrow (e^x - 2)(e^x + 1) = 0$

$\Rightarrow\; e^x = 2$ not possible.

(C) $x = \pm 4$ are two integral solutions.

(D) $-1 \leq |x - 1| - 1 \leq 1 \;\Rightarrow\; 0 \leq |x - 1| \leq 2$

$-2 \leq x - 1 \leq 2 \;\Rightarrow\; -1 \leq x \leq 3$

$\Rightarrow\; x = -1, 0, 1, 2, 3$

There are 5 solutions.

41. (A) → (p, s),(B) → (q, s), (C) → (q, s, t), (D) → (p, q, r, s,t)

(A) $\sin^{-1}[x - 1] + \sec^{-1}[x - 1]$

$-1 \leq [x - 1] \leq 1\,;\; -1 \leq x - 1 < 2$

$\Rightarrow \quad 0 \leq x < 3 \qquad \text{... (i)}$

Also $[x - 1] \leq -1$ or $[x - 1] \geq 1$

$\Rightarrow \quad x - 1 < 0$ or $x \geq 2$

$\Rightarrow \quad x < 1$ or $x \geq 2 \qquad \text{... (ii)}$

From (i) and (ii), $x \in [0, 1) \cup [2, 3)$

(B) $\sqrt{(x+2)(5-x)}-\frac{1}{\sqrt{x^2-4}}$

$(x + 2)(5 - x) \geq 0 \;\Rightarrow\; x \in [-2, 5] \qquad \text{.... (i)}$

Also, $x^2 - 4 > 0 \Rightarrow x \in (-\infty, -2) \cup (2, \infty) \qquad \text{.... (ii)}$

From (i) and (ii), $x \in (2, 5]$

(C) $\log_{10}(1 + x^3) > 0$

$1 + x^3 > 1 \;\Rightarrow x^3 > 0 \;\Rightarrow x > 0 \;\Rightarrow x \in (0, \infty)$

(D) $y = \log_e(x^2 - x + 1) \Rightarrow x^2 - x + 1 > 0 \Rightarrow x \in R$ as $D < 0$

42. (A) → (s), (B) → (r), (C) → (r), (D) → (p)

(A) $(gof)\left(\frac{1}{2}\right)=g\left[f\left(\frac{1}{2}\right)\right]=g\left(\frac{5}{4}\right)=2(1)-1=1$

(B) $(fog)\left(\frac{3}{2}\right)=f\left[g\left(\frac{3}{2}\right)\right]=f(1)=1^2+1=2$

(C) $(fogof)\left(\frac{3}{4}\right)=(fog)\left(f\left(\frac{3}{4}\right)\right)=(fog)\left(\frac{25}{16}\right)=f\left[g\left(\frac{25}{16}\right)\right]$

$= f(1) = 1^2 + 1 = 2$

(D) $(gofog)\left(\frac{2}{3}\right)=(gof)\left[g\left(\frac{2}{3}\right)\right]=(gof)(-1)=g[f(-1)]$

$= g(2) = 2 \times 2 - 1 = 3$

43. (4) : $f(x + y) - kxy = f(x) + 2y^2$

Put $x = 0 \;\Rightarrow\; f(y) = f(0) + 2y^2 \;\Rightarrow\; f(x) = 2x^2 + f(0)$

Now $f(1) = 2 \;\Rightarrow\; f(0) = 0$

$\therefore \quad f(x) = 2x^2$

Hence $f(x+y)\times f\left(\frac{1}{x+y}\right)=4$

44. (7) : Given,

$$f(x-f(y))=f(f(y))+xf(y)+f(x)-1 \qquad \text{...(1)}$$

Putting $x = 0$ and $f(y) = 0$, we get

$$f(0)=f(0)+f(0)-1 \;\Rightarrow\; f(0)=1 \qquad \text{...(2)}$$

Putting $x = a$ and $f(y) = a$, we get

$$f(0) = f(a) + a^2 + f(a) - 1$$

$$\Rightarrow \quad 1 = a^2 + 2f(a) - 1 \quad \Rightarrow \quad f(a) = 1 - \frac{a^2}{2}$$

$$\Rightarrow \quad f(x) = 1 - \frac{x^2}{2} \qquad \text{[Putting } x \text{ in place of } a\text{]}$$

$$\therefore \quad -f(10) = -1 + \frac{10^2}{2} = 49$$

45. (4) : If $f(x) = \dfrac{ax+b}{cx-a}, x \neq \dfrac{a}{c}$, then $f(f(x)) = x$

$$\therefore \quad f(f(x)) = x \text{ and } f\left(f\left(\frac{4}{x}\right)\right) = \frac{4}{x}$$

$$\therefore \quad f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right) = x + \frac{4}{x} \geq 4$$

46. (9) : $f[f(x)] = x \Rightarrow d = -a \quad \therefore \quad f(x) = \dfrac{ax+b}{cx-a}$

$$f(5) = 5 \quad \Rightarrow \quad 5a + b = (5c - a)\cdot 5 = 25c - 5a \qquad \text{...(1)}$$

$$f(13) = 13 \quad \Rightarrow \quad 13a + b = (13c - a)\cdot 13 = 169c - 13a \qquad \text{...(2)}$$

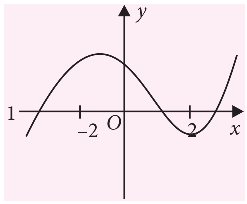
(1) – (2), we get, $-8a = -144c + 8a \Rightarrow a = 9c$

$$\therefore \quad f(x) = \frac{9cx + b}{cx - a}$$

Range $f(x)$ does not contain $\dfrac{9c}{c} = 9 \Rightarrow x = 9$

47. (9) : $f(x) = x^3 - 12x + P$

$\Rightarrow f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$



$f(x)$ increases in $(-\infty, -2) \cup (2, \infty)$

$f(x)$ decreases in $(-2, 2)$

$\because$ $f(x)$ is continuous

$$\left.\begin{array}{l} f(-2) = P + 16 \\ f(2) = P - 16 \end{array}\right\} \forall P \in \{1, 2, \ldots\ldots, 15\}$$

$f(-2) > 0, \ f(2) < 0$

If has 3 roots in each case $\therefore \ \Sigma Q = 3 \times 15 = 45$

48. (1) : $f(2011) = (2 + 0 + 1 + 1)^2 = 16$

$f^2(2011) = (1 + 6)^2 = 49, f^3(2011) = (4 + 9)^2 = 169$

$f^4(2011) = (1 + 6 + 9)^2 = 256, f^5(2011) = (2 + 5 + 6)^2 = 169$

It is a periodic function.

$\Rightarrow \ f^{2n}(2011) = 256$ and $f^{2n+1}(2011) = 169 \ \forall \ n \geq 2$

49. (2) : Putting $\tan x = t$, we get $f(t)$ is a polynomial function of the form, $f(t) = \pm t^n + 1$

When $t = 2, f(2) = 9 \ \Rightarrow \ n = 3 \therefore f(t) = t^3 + 1 \Rightarrow f'(t) = 3t^2$

$$f'(2) = 12 \quad \therefore \quad \frac{f'(2)}{6} = 2$$

50. (5) : $2 + f(x) \cdot f(y) = f(x) + f(y) + f(xy) \ \forall \ x, y \in R$ ...(i)

$f(2) = 5$

$$\text{Put } y = \frac{1}{x} \Rightarrow 2 + f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1) \quad \text{...(ii)}$$

Put $x = 1 \ \Rightarrow \ 2 + f(1) \cdot f(1) = f(1) + f(1) + f(1)$

$\Rightarrow \ [f(1)]^2 - 3 \cdot f(1) + 2 = 0 \Rightarrow [f(1) - 2][f(1) - 1] = 0$

As $f(2) = 5 \ \Rightarrow \ f(1) \neq 1 \ \Rightarrow \ f(1) = 2$

$$\text{(ii)} \quad \Rightarrow \quad 2 + f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + 2$$

$\Rightarrow \ f(x) = \pm x^n + 1$ Since $f(2) = 5 \ \Rightarrow \ n = 2 \ \Rightarrow \ f(x) = x^2 + 1$

$\therefore \ f(f(1)) = f(2) = 5$

CLASS XII **Series 4**

ACE YOUR WAY CBSE

Continuity and Differentiability | Application of Derivatives

	Previous Years Analysis					
	2016		2015		2014	
	Delhi	AI	Delhi	AI	Delhi	AI
VSA	-	-	-	-	-	-
SA	3	3	3	3	3	3
LA	1	1	1	1	1	1

HIGHLIGHTS

CONTINUITY

	Continuity	Discontinuity
At a point	• A function $f(x)$ is said to be continuous at a point $x = a$ in the domain of $f(x)$, if (i) $f(a)$ is defined (ii) $\lim\limits_{x\to a} f(x)$ exists and (iii) $\lim\limits_{x\to a} f(x) = f(a)$ • A function $f(x)$ is said to be continuous at a point $x = a$ in its domain, if $f(a)$ exists, $\lim\limits_{x\to a^-} f(x)$ exists, $\lim\limits_{x\to a^+} f(x)$ exists and $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a)$. The point where the function is continuous is called the point of continuity.	• If the function $f(x)$ is not continuous at $x = a$, it is said to be discontinuous at $x = a$. Here 'a' is called the point of discontinuity. • The discontinuity of a function $f(x)$ at $x = a$ may be due to any of the following reasons: (i) $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ or, both may not exist. (ii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist but are not equal. (iii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist and are equal but both may not be equal to $f(a)$.
In an interval	**Continuity on an Open Interval** A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at each point of (a, b). **Continuity on a Closed Interval** A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if (i) $f(x)$ is continuous from right at $x = a$, *i.e.* $\lim\limits_{h\to 0} f(a+h) = f(a)$ (ii) $f(x)$ is continuous from left at $x = b$, *i.e.* $\lim\limits_{h\to 0} f(b-h) = f(b)$ (iii) $f(x)$ is continuous at each point of the open interval (a, b).	A real valued function $f(x)$ is said to be discontinuous if it is not continuous at atleast one point in the given interval.

Note :

- Sum, difference and product of two continuous functions are continuous.
- If $f(x)$ and $g(x)$ are two continuous functions, then $\frac{f(x)}{g(x)}$ (provided $g(x) \neq 0$), is continuous.

DIFFERENTIABILITY

Left Hand Derivative

If $y = f(x)$ is a real valued function and a is any real number, then $\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the left hand derivative of $f(x)$ at $x = a$ and is denoted by $Lf'(a)$.

Right Hand Derivative

If $y = f(x)$ is a real valued function and a is any real number, then $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the right hand derivative of $f(x)$ at $x = a$ and is denoted by $Rf'(a)$.

Differentiability

A real valued function $f(x)$, is said to be differentiable at a point $x = a$ if and only if

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ or } Lf'(a) = Rf'(a)$$

Note :

- Every differentiable functions is continuous but the converse is not necessarily true.
- A real valued function $f(x)$ is not differentiable
  (i) at a point $x = a$ if $Lf'(a) \neq Rf'(a)$.
  (ii) in an interval if it is not differentiable atleast one point in the interval.

SOME PROPERTIES OF DERIVATIVES

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.
6.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
7.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
8.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$

SOME GENERAL DERIVATIVES

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec}x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$; $x\in(-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$; $x\in(-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2}$; $x\in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$; $x \in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$; $x \in R-[-1, 1]$	$\operatorname{cosec}^{-1}x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$; $x \in R-[-1,1]$
$\log_e x$	$\frac{1}{x}$; $x > 0$	a^x	$a^x \log_e a$; $a > 0$	$\log_a x$	$\frac{1}{x\log_e a}$; $x > 0$ and $a > 0$

Important Theorems

Rolle's Theorem

If a real valued function $f(x)$
(i) is continuous in $[a, b]$
(ii) is differentiable on (a, b)
(iii) $f(a) = f(b)$,
then there exist at least one real number c in the interval (a, b) such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

If a real valued function $f(x)$ is
(i) continuous in $[a, b]$
(ii) differentiable on (a, b),
then there exists at least one real number c, where $a<c<b$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

APPLICATION OF DERIVATIVES

RATE OF CHANGE OF QUANTITIES

Let $y = f(x)$ be a function. Then $\frac{dy}{dx}$ denotes the rate of change of y w.r.t. x.

INCREASING AND DECREASING FUNCTIONS

A function $f(x)$ defined on (a, b) is said to be

		Using derivative test
Increasing Function	If $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) \ge 0$ for each $x \in (a, b)$.
Strictly Increasing Function	If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) > 0$ for each $x \in (a, b)$.
Decreasing Function	If $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) \le 0$ for each $x \in (a, b)$.
Strictly Decreasing Function	If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) < 0$ for each $x \in (a, b)$.

TANGENTS AND NORMALS

Let $y = f(x)$ be a curve. Then, slope and equation of tangent and normal at (x_1, y_1) is given by,

	Tangent	Normal
Slope	$\left.\dfrac{dy}{dx}\right\|_{(x_1,y_1)}$	$\dfrac{-1}{\left.\left(\dfrac{dy}{dx}\right)\right\|_{(x_1,y_1)}}$
Equations	$y-y_1=\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}(x-x_1)$	$y-y_1=\dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}}(x-x_1)$

Remarks

- If $\dfrac{dy}{dx}=0$ at (x_1, y_1), then tangent is parallel to x-axis. Hence, equation of tangent is $y = y_1$.
- If $\dfrac{dy}{dx}=\infty$ at (x_1, y_1), then tangent is perpendicular to x-axis. Hence, equation of tangent is $x = x_1$.
- Angle between two curves is $\tan\theta=\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$, where m_1 and m_2 are slopes of tangents at the point of intersection of curve.

APPROXIMATIONS

Let $y = f(x)$ be a differentiable function. Δx and Δy be the small changes in x and y respectively. Then, we find the approximate value of certain quantity by the following steps :

I. Find Δx and x

II. $\Delta y=\dfrac{dy}{dx}(\Delta x)$

III. $\Delta y = f(x + \Delta x) - f(x)$ or $f(x + \Delta x) = f(x) + \Delta y$

MAXIMA AND MINIMA

Maximum value of $f(x)$: Let $f(x)$ be a function defined on an interval I. Then f is said to have maximum value in I, if there exists a point $a \in I$, such that $f(a) \geq f(x)$ $\forall\ x \in I$.

Minimum value of $f(x)$: Let $f(x)$ be a function defined on an interval I. Then f is said to have minimum value in I, if there exists a point $a \in I$, such that $f(a) \leq f(x)$ $\forall\ x \in I$.

LOCAL MAXIMA AND LOCAL MINIMA

To find the local maximum or local minimum values of a function, the following tests are useful.

I. First derivative test : A function $f(x)$ is said to have a local maximum value at $x = a$, if

(i) $f'(a) = 0$

(ii) $f'(a - h) > 0$

(iii) $f'(a + h) < 0$, where h is small positive number.

A function $f(x)$ is said to have a local minimum value at $x = a$ if

(i) $f'(a) = 0$

(ii) $f'(a - h) < 0$

(iii) $f'(a + h) > 0$, where h is small positive number.

Note : $f(a)$ is neither local maximum nor minimum, if $f'(a) = 0$ and $f'(a - h) > 0$, $f'(a + h) > 0$ or $f'(a - h) < 0$, $f'(a + h) < 0$. In this case, $x = a$ is called the point of inflection.

II. Second derivative test : A function $y = f(x)$ is said to have a local maximum value at $x = a$ if $f'(a) = 0$ and $f''(a) < 0$ and a local minimum value at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$.

Note : If $f''(a) = 0$, then second derivative test fails.

PROBLEMS

Very Short Answer Type

1. Show that $f(x) = x^3$ is continuous at $x = 2$.
2. If $y = x^4 + 10$ and x changes from 2 to 1.99, find the approximate change in y.
3. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing on $(-1, 1)$.
4. The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 40x + 10$ Find the marginal revenue when $x = 5$.
5. Find the derivative of $(4x^3 - 5x^2 + 1)^4$ w.r.t. to x.

Long Answer Type-I

6. Verify Rolle's Theorem for the function $f(x) = (x - 1)(x - 2)^2$ in [1,2].

7. Find the intervals in which the function $f(x)=\dfrac{4x^2+1}{x}(x\neq 0)$ is
   (i) increasing (ii) decreasing

8. For what choices of a and b, the function $f(x)=\begin{cases} x^2, & x\le c\\ ax+b, & x>c\end{cases}$ is differentiable at $x = c$.

9. Differentiate $(e^x \cos^3 x \sin^2 x)$ w.r.t. x.

10. An open box is to be made out of a piece of cardboard measuring (24 cm × 24 cm) by cutting off equal squares from the corners and turning up the sides. Find the height of the box when it has maximum volume.

Long Answer Type-II

11. Find the equations of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$. Also, find the point of intersection where both tangent and normal cut the x-axis.

12. If $f(x)=\dfrac{\sqrt{2}\cos x-1}{\cot x-1}, x\neq\dfrac{\pi}{4}$, find the value of $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at $x=\dfrac{\pi}{4}$.

13. Prove that the height and the radius of the base of an open cylinder of given surface area and maximum volume are equal.

14. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of a and b.

15. Verify Lagrange's mean value theorem for the function $f(x) = x(x - 1)(x - 2)$ in the interval $\left[0,\dfrac{1}{2}\right]$.

SOLUTIONS

1. We have, $f(2) = 2^3 = 8$;

$$\lim_{x\to 2^+} f(x)=\lim_{h\to 0}(2+h)^3=\lim_{h\to 0}(8+h^3+12h+6h^2)=8;$$

$$\lim_{x\to 2^-} f(x)=\lim_{h\to 0}(2-h)^3=\lim_{h\to 0}(8-h^3-12h+6h^2)=8$$

$$\because \lim_{x\to 2^+} f(x)=\lim_{x\to 2^-} f(x)=f(2)$$

Hence, $f(x)$ is continuous at $x = 2$.

2. Given $y = x^4 + 10$ ...(i)

$\therefore \dfrac{dy}{dx}=4x^3$ ...(ii)

Approximate change in y is given by

$\Delta y=\dfrac{dy}{dx}\Delta x=4x^2\Delta x$ ... (iii)

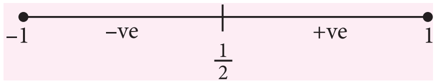
Putting $x = 2$ and $x + \Delta x = 1.99 \Rightarrow \Delta x = -0.01$ in (iii), we get

$\Delta y = 4(2)^2 \cdot (-0.01) = -0.16$

3. Given, $f(x) = x^2 - x + 1$

$\therefore$ $f'(x) = 2x - 1$

Sign scheme for $f'(x)$ in $(-1, 1)$ is



Since $f'(x)$ changes sign in $(-1, 1)$, therefore, it is neither increasing nor decreasing in $(-1, 1)$.

4. We have, $R(x) = 3x^2 + 40x + 10$

$\Rightarrow$ $\text{MR}=\dfrac{dR}{dx}=\dfrac{d}{dx}(3x^2+40x+10)=6x+40$

$\Rightarrow$ $[\text{MR}]_{x=5} = (6 \times 5 + 40) = 70$

Hence, the required marginal revenue is ₹70.

5. $y = u^4$, where $u = 4x^3 - 5x^2 + 1$

Now, $\dfrac{dy}{du}=4u^3$ and $\dfrac{du}{dx}=12x^2-10x$

By chain rule, $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$\therefore \dfrac{dy}{dx}=4u^3(12x^2-10x)$

$= 4(4x^3 - 5x^2 + 1)^3(12x^2 - 10x)$

6. The given function is

$f(x) = (x - 1)(x - 2)^2, x \in [1,2]$

Clearly $f(x)$ is a polynomial in x, therefore, $f(x)$ is continuous and differentiable everywhere.

$\therefore$ $f(x)$ is continuous on [1, 2]

and $f(x)$ is differentiable on (1, 2)

Also $f(1) = 0$ and $f(2) = 0$

Hence all conditions of Rolle's theorem are satisfied for $f(x)$ in [1, 2]

$\Rightarrow$ $\exists\, c \in (1, 2)$ satisfying $f'(c) = 0$

Now $f'(x) = 1\cdot(x - 2)^2 + (x - 1)\cdot 2(x - 2)$

$= (x - 2)(x - 2 + 2x - 2)$

$= (x - 2)(3x - 4)$

$\because \quad f'(c)=0 \Rightarrow c=2$ or $c=\frac{4}{3}$

But $c = 2 \notin]1, 2[$, therefore, $c=\frac{4}{3}\in]1,2[$ satisfying $f'(c) = 0$

Thus, Rolle's theorem has been verified.

7. We have, $f(x)=\frac{4x^2+1}{x}$, $(x \neq 0)$

Differentiating w.r.t. to x, we get

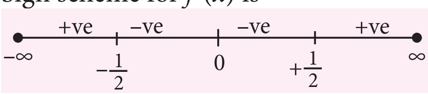
$$f'(x)=4-\frac{1}{x^2}=\frac{4x^2-1}{x^2}$$

For $f'(x) = 0 \Rightarrow 4x^2 - 1 = 0$

$\Rightarrow \quad (2x - 1)(2x + 1) = 0$

$\Rightarrow \quad x=\frac{1}{2}$ or $x=\frac{-1}{2}$

Sign scheme for $f'(x)$ is



(i) Now for $f(x)$ to be increasing, $f'(x) \geq 0$

$\therefore \quad f(x)$ is increasing in $\left(-\infty,-\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

Since, $f(x)$ is continuous at $x=\frac{-1}{2}$ and $x=\frac{1}{2}$.

$\therefore \quad f(x)$ is increasing in $\left(-\infty,\frac{-1}{2}\right]\cup\left[\frac{1}{2},\infty\right)$

(ii) For $f(x)$ to be decreasing, $f'(x) \leq 0$

$\therefore \quad f(x)$ is decreasing in $\left(-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right)$

Since, $f(x)$ is continuous at $x=-\frac{1}{2}$ and $x=\frac{1}{2}$

$\therefore \quad f(x)$ is decreasing in $\left[-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right]$

Here, 0 is excluded as $f(x)$ is not defined at $x = 0$

8. It is given that $f(x)$ is differentiable at $x = c$ and every differentiable function is continuous. So, $f(x)$ is continuous at $x = c$.

$\therefore \quad \lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=f(c)$

$\Rightarrow \quad \lim_{x\to c^-} (x^2)=\lim_{x\to c^+} (ax+b)=c^2$

$\Rightarrow \quad c^2 = ac + b$ ... (i)

Now, $f(x)$ is differentiable at $x = c$

$\therefore \quad Lf'(c) = Rf'(c)$

$\Rightarrow \quad \lim_{x\to c^-} = \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$

$\Rightarrow \quad \lim_{x\to c^-} \frac{x^2-c^2}{x-c} = \lim_{x\to c^+} \frac{(ax+b)-c^2}{x-c}$

$\Rightarrow \quad \lim_{x\to c}(x+c)=\lim_{x\to c}\frac{ax+b-(ac+b)}{x-c}$ [From (i)]

$\Rightarrow \quad \lim_{x\to c}(x+c)=\lim_{x\to c}\frac{a(x-c)}{x-c}$

$\Rightarrow \quad \lim_{x\to c}(x+c)=\lim_{x\to c}(a) \Rightarrow 2c = a$ ... (ii)

From (i) and (ii) we get

$a = 2c$ and $b = -c^2$

9. Let $y = e^x\cos^3 x\sin^2 x$

Taking log on both sides, we get

$\log y = x + 3 \log \cos x + 2\log \sin x$ ...(i)

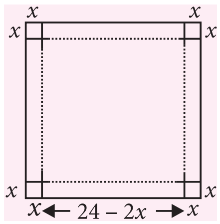
Differentiating (i) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=1+\frac{3}{\cos x}\cdot(-\sin x)+\frac{2}{\sin x}\cdot\cos x$$

$\Rightarrow \quad \frac{dy}{dx}=y.\{1-3\tan x+2\cot x\}$

$= (e^x\cos^3 x\sin^2 x)(1 - 3\tan x + 2\cot x)$.

10. Let the length of the side of the each square cut off from the corners be x cm.



Then, height of the box = x cm.

Volume $V = (24 - 2x)^2 \times x$

$= 4x^3 - 96x^2 + 576x$

$\Rightarrow \quad \frac{dV}{dx}=12(x^2-16x+48)$

and $\frac{d^2V}{dx^2}=24(x-8)$

Now, for maximum or minimum volume,

$\frac{dV}{dx}=0 \Rightarrow x^2-16x+48=0$

$\Rightarrow \quad (x - 12)(x - 4) = 0 \Rightarrow x = 4$ [$\because x \neq 12$]

$\left[\frac{d^2V}{dx^2}\right]_{x=4}=-96<0$

$\therefore \quad V$ is maximum at $x = 4$.

So, required height = 4 cm.

11. Given curve is $16x^2 + 9y^2 = 144$ ... (i)

Differentiating w.r.t. x, we get

$32x+18y\frac{dy}{dx}=0$

$\Rightarrow \quad \frac{dy}{dx}=-\frac{16}{9}\cdot\frac{x}{y}$ ... (ii)

Now, $P\,(2, y_1)$ lies on (i), therefore,

$$16 \times 4 + 9y_1^2 = 144$$

$$\Rightarrow \quad 9y_1^2 = 80 \Rightarrow \quad y_1 = \frac{4\sqrt{5}}{3} \qquad [\because y_1 > 0]$$

$$\therefore \quad P \equiv \left(2, \frac{4\sqrt{5}}{3}\right)$$

At point P, $\dfrac{dy}{dx} = -\dfrac{16}{9} \times \dfrac{2}{\frac{4\sqrt{5}}{3}} = -\dfrac{8}{3\sqrt{5}}$

$\therefore$ Equation of the tangent at P is

$$y - \frac{4\sqrt{5}}{3} = -\frac{8}{3\sqrt{5}}(x-2)$$

This meets x-axis where $y = 0$

$$\therefore \quad 0 - \frac{4\sqrt{5}}{3} = -\frac{8}{3\sqrt{5}}(x-2)$$

$$\Rightarrow \quad \frac{5}{2} = x - 2 \text{ or } x = \frac{9}{2}$$

$\therefore$ Tangent meets x-axis at $\left(\dfrac{9}{2}, 0\right)$

Also equation of normal at P is

$$y - \frac{4\sqrt{5}}{3} = \frac{3\sqrt{5}}{8}(x-2)$$

This meet x-axis where $y = 0$

$$\therefore \quad 0 - \frac{4\sqrt{5}}{3} = \frac{3\sqrt{5}}{8}(x-2)$$

$$\text{or} \quad -\frac{32}{9} = x - 2 \text{ or } x = -\frac{14}{9}$$

$\therefore$ Normal meets x-axis at $\left(-\dfrac{14}{9}, 0\right)$.

12. Given, $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$

$$\therefore \quad \lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

Putting $x = \dfrac{\pi}{4} + h$ so that as $x \to \dfrac{\pi}{4}, h \to 0$

$$= \lim_{h \to 0} \frac{\sqrt{2}\cos\left(\frac{\pi}{4} + h\right) - 1}{\cot\left(\frac{\pi}{4} + h\right) - 1}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\left(\cos\frac{\pi}{4}\cos h - \sin\frac{\pi}{4}\sin h\right) - 1}{\frac{1-\tan h}{1+\tan h} - 1}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos h - \frac{1}{\sqrt{2}}\sin h\right) - 1}{-2\tan h}(1+\tan h)$$

$$= \lim_{h \to 0} \frac{\cos h - \sin h - 1}{-2\tan h}(1+\tan h)$$

$$= \lim_{h \to 0} \frac{1 - \cos h + \sin h}{2\tan h}(1+\tan h)$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{h}{2} + 2\sin\frac{h}{2}\cos\frac{h}{2}}{2\tan h}(1+\tan h)$$

$$= \lim_{h \to 0} \frac{\sin\frac{h}{2}\left(\sin\frac{h}{2} + \cos\frac{h}{2}\right)}{\tan h}(1+\tan h)$$

$$= \lim_{h \to 0} \frac{\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\frac{h}{2}\left(\sin\frac{h}{2} + \cos\frac{h}{2}\right)}{\left(\frac{\tan h}{h}\right)\cdot h}(1+\tan h)$$

$$= \frac{1}{2}(0+1)(1+0) = \frac{1}{2} \quad \therefore \quad \lim_{x \to \frac{\pi}{4}} f(x) = \frac{1}{2}$$

Hence, for $f(x)$ to be continuous at $x = \dfrac{\pi}{4}, f\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}$

13. Let x be the radius of the cylinder and z be its height, let S be the area of the surface of the cylinder and V be its volume.

Given S = constant

Now, $S = 2\pi xz + \pi x^2$ ... (i)

and $V = \pi x^2 z$ ...(ii)

From (i), $2\pi xz = S - \pi x^2$

$$\therefore \quad z = \frac{S - \pi x^2}{2\pi x} \qquad \text{...(iii)}$$

From (ii), $V = \pi x^2\left(\dfrac{S - \pi x^2}{2\pi x}\right) = \dfrac{Sx}{2} - \dfrac{\pi x^3}{2}$

$$\therefore \quad \frac{dV}{dx} = \frac{S}{2} - \frac{3\pi x^2}{2}$$

For maximum or minimum values of V,

$$\frac{dV}{dx} = 0 \Rightarrow \frac{S}{2} - \frac{3\pi}{2}x^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{S}{3\pi} \quad \Rightarrow \quad x = \sqrt{\frac{S}{3\pi}}$$

Now, $\dfrac{d^2V}{dx^2}=-\dfrac{3\pi}{2}\cdot 2x=-3\pi x$

When $x=\sqrt{\dfrac{S}{3\pi}},\dfrac{d^2V}{dx^2}=-3\pi\sqrt{\dfrac{S}{3\pi}}=-\sqrt{3\pi S}<0$

Hence when $x=\sqrt{\dfrac{S}{3\pi}}$, V is maximum.

Now, $\dfrac{x}{z}=\dfrac{2\pi x^2}{5-\pi x^2}\Rightarrow\dfrac{x}{z}=\dfrac{2\pi\cdot\dfrac{S}{3\pi}}{S-\pi\dfrac{S}{3\pi}}=\dfrac{\frac{2}{3}S}{\frac{2}{3}S}=1\Rightarrow x=z$

Thus for maximum volume of a cylinder to given surface area $x = z$.

14. Given, $(x-a)^2+(y-b)^2=c^2$ ...(i)

Differentiating both sides w.r.t. x, we get

$$2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow\quad \frac{dy}{dx}=-\frac{x-a}{y-b} \quad\text{... (ii)}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=-\frac{(y-b).1-(x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$=-\frac{(y-b)-(x-a)\left\{-\left(\frac{x-a}{y-b}\right)\right\}}{(y-b)^2} \quad\text{[From (ii)]}$$

$$=-\frac{\frac{(x-a)^2}{y-b}+(y-b)}{(y-b)^2}=-\frac{(x-a)^2+(y-b)^2}{(y-b)^3}$$

$$=-\frac{c^2}{(y-b)^3} \quad\text{...(iii)}$$

Now, $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}=\left[1+\dfrac{(x-a)^2}{(y-b)^2}\right]^{3/2}$

$$=\left[\frac{(x-a)^2+(y-b)^2}{(y-b)^2}\right]^{3/2}\frac{(c^2)^{3/2}}{(y-b)^3}=\frac{c^3}{(y-b)^3} \quad\text{...(iv)}$$

Now from (iii) and (iv), we have

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=-c$$

which is a constant independent of a and b.

15. Given $f(x) = x(x-1)(x-2) = x(x^2-3x+2)$

$\therefore\quad f'(x)=1\cdot(x^2-3x+2)+x(2x-3)$
$\qquad\qquad = 3x^2-6x+2$

Clearly $f(x)$ is a polynomial in x, therefore, it is continuous and differentiable for all x

Hence, $f(x)$ is continuous in $\left[0,\dfrac{1}{2}\right]$

and $f(x)$ is differentiable in $\left(0,\dfrac{1}{2}\right)$

Now $f(0) = 0$ and $f\left(\dfrac{1}{2}\right)=\dfrac{1}{2}\left(\dfrac{1}{2}-1\right)\left(\dfrac{1}{2}-2\right)=\dfrac{3}{8}$

Hence, all conditions of Lagrange's mean value theorem are satisfied for $f(x)$ in $\left[0,\dfrac{1}{2}\right]$.

$\therefore$ There exists at least one $c\in\left[0,\dfrac{1}{2}\right]$ such that

$$f'(c)=\frac{f\left(\frac{1}{2}\right)-f(0)}{\frac{1}{2}-0}$$

$$\Rightarrow\quad 3c^2-6c+2=\frac{\frac{3}{8}-0}{\frac{1}{2}}=\frac{3}{4}\Rightarrow 12c^2-24c+5=0$$

$$\therefore\quad c=\frac{24\pm\sqrt{576-240}}{24}=\frac{24\pm\sqrt{336}}{24}$$

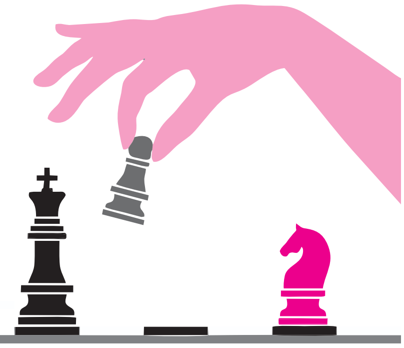
$$=\frac{24\pm4\sqrt{21}}{24}=1\pm\frac{\sqrt{21}}{6}$$

Hence, $c=1+\dfrac{\sqrt{21}}{6},1-\dfrac{\sqrt{21}}{6}$

But $0<c<\dfrac{1}{2}\quad\therefore\ c=1-\dfrac{\sqrt{21}}{6}$

Thus Lagrange's mean value theorem has been verified.

Challenging PROBLEMS ON CALCULUS



SECTION-I

Single Correct Answer Type

1. For positive a and b, define $Y(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$, then for $0 < a < 1$, $\int_0^{\infty}\frac{y^{a-1}}{1+y}dy=$
   (a) $Y\left(a,\frac{1}{a}\right)$ (b) $Y(a, 1-a)$
   (c) $Y(1-a, a)$ (c) $Y(a, 1+a)$

2. Let $f(x)=\frac{\sin x}{x}$, then $\int_0^{\pi/2} f(x)f\left(\frac{\pi}{2}-x\right)dx=$
   (a) $\pi\int_0^{\pi} f(x)dx$ (b) $\frac{1}{\pi}\int_0^{\pi} f(x)dx$
   (c) $\frac{2}{\pi}\int_0^{\pi} f(x)dx$ (d) $\frac{\pi}{2}\int_0^{\pi} f(x)dx$

3. $\int\frac{(2x+3)dx}{x(x+1)(x+2)(x+3)+1}=\frac{-1}{f(x)}+c$ where $f(x)$ is
   (a) linear (b) quadratic
   (c) cubic (d) quartic

4. $\int_0^1\frac{\tan^{-1}ax}{x\sqrt{1-x^2}}dx=\lambda\pi\log(a+\sqrt{1+a^2})$ for $\lambda=$
   (a) $\frac{1}{2}$ (b) 1 (c) 2 (d) $\frac{1}{3}$

5. Let $I_1=\int_0^{\pi/2}\sin^{\sqrt{2}+1}x\,dx$, $I_2=\int_0^{\pi/2}\sin^{\sqrt{2}-1}x\,dx$, then $\frac{I_1}{I_2}=$
   (a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$
   (c) $4+2\sqrt{2}$ (d) $4-2\sqrt{2}$

6. If a, b, c are positive constants $(a > b)$ then the max. value of r, given by $\frac{c^4}{r^2}=\frac{a^2}{\sin^2\theta}+\frac{b^2}{\cos^2\theta}$ is
   (a) $\frac{c^2}{\sqrt{ab}}$ (b) $\frac{c^2}{a-b}$ (c) $\frac{c^2}{a+b}$ (d) $\frac{c}{a-b}$

7. Let N be any four digit number say $(a_1\,a_2a_3a_4)$, then the maximum value of $\frac{N}{a_1+a_2+a_3+a_4}=$
   (a) 1001 (b) 1111 (c) 1000 (d) 900

SECTION-II

Multiple Correct Answer Type

8. Let $y_1=|\sin||x|-\pi||$ and $y_2=|\sin x|$ be two curves, $x\in R$ then
   (a) area of region bounded by the two curves with x-axis is same for all intervals (a, b)
   (b) $\left|\int_a^b y_1dx\right|>\left|\int_a^b y_2dx\right|$
   (c) y_1 is periodic with period π
   (d) y_2 is periodic with period π

9. $\int_0^{\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$; then
   (a) $\int_0^{\infty}e^{-2x^2}dx=\frac{\sqrt{\pi}}{\sqrt{2}}$ (b) $\int_0^{\infty}xe^{-x^2}dx=\frac{1}{2}$
   (c) $\int_0^{\infty}x^2e^{-x^2}dx=\frac{\pi}{4}$ (d) $\int_0^{\infty}x^2e^{-x^2}dx=\frac{\sqrt{\pi}}{4}$

10. If $I=\int_0^{\infty}\frac{\sin x}{x}dx$, then $\int_0^{\infty}\frac{\sin ax\cos bx}{x}=(a,b>0)$
    (a) I if $a > b$ (b) $2I$ if $a + b = 0$
    (c) $\frac{I}{a+b}$ (d) 0 if $a < b$

11. Let $I=\lim_{n\to\infty}\int_a^{\infty}\frac{ndx}{1+n^2x^2}$, $a\in R$ then I can be
    (a) 0 (b) 1 (c) $\pi/2$ (d) π

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SECTION-III

Comprehension Type

Paragraph for Question No. 12 to 13

A function f satisfies the equation $f(x)f'(-x) = f(-x)f'(x)$ for all x and $f(0) = 3$

12. The value of $f(x) \cdot f(-x)$ for all x, is

(a) 1 (b) 4 (c) 9 (d) 16

13. $\int_{-51}^{51} \frac{dx}{3+f(x)} =$

(a) 17 (b) 34 (c) 102 (d) 0

Paragraph for Question No. 14 to 15

Suppose a and b are positive real numbers such that $ab = 1$. Let for any real parameter t, the distance from the origin to the line $(ae^t)x + (be^{-t})y = 1$ be denoted by $D(t)$, then

14. The value of 'b' at which $I = \int_0^1 \frac{dt}{(D(t))^2}$ is minimum, is

(a) $\sqrt{e}$ (b) e (c) $\frac{1}{\sqrt{e}}$ (d) $\frac{1}{e}$

15. Minimum value of I is

(a) $e+\frac{1}{e}$ (b) $e-\frac{1}{e}$ (c) $e - 1$ (d) e

SECTION-IV

Integer Answer Type

16. If $\int_0^{2105\pi} e^{\cos\theta} \cdot \cos(\sin\theta)d\theta = k\pi$, then $\left[\frac{k}{1000}\right] =$

([·] denotes G.I.F)

17. If $\frac{d^3x}{dy^3} = \frac{k\left(\frac{d^2y}{dx^2}\right)^2 - \frac{d^3y}{dx^3}\cdot\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)^5}$, then $k =$

18. If the value of

$$\int \frac{1-(\cot x)^8}{\tan x + (\cot x)^9} dx = \frac{1}{k}\log|\sin^k x + \cos^k x| + c$$

then the unit's digit of k is

19. If $f(x^2+x)(x^{-8}+2x^{-9})^{1/10} = \frac{5}{11}(f(x))^{\lambda/\mu} + c$, then $\lambda - \mu =$

20. If $I = \int_0^1 x^{70}(1-x)^{30}dx$, then $101 \times {}^{100}C_{30} \times I =$

SOLUTIONS

1. (b) : Substitute $x = \frac{y}{1+y}$ in $Y(a, b)$.

2. (c): Given integral is

$$I = \int_0^{\pi/2} \frac{\sin 2x}{x(\pi-2x)}dx = \int_0^{\pi} \frac{\sin t}{t(\pi-t)}dt\left(\text{put } x = \frac{t}{2} \Rightarrow dx = \frac{dt}{2}\right)$$

or $I = \frac{1}{\pi}\cdot\int_0^{\pi}\left(\frac{\sin t}{t} + \frac{\sin t}{\pi - t}\right)dt$

Hence, $I = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt = \frac{2}{\pi}\int_0^{\pi}\frac{\sin x}{x}dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

3. (b) : Simplify the denominator and put $x^2 + 3x = t$ to get $I = \int \frac{dt}{t(t+2)+1} = \int \frac{dt}{(t+1)^2}$

4. (a) : Let $I = \int_0^1 \frac{\tan^{-1} ax}{x\sqrt{1-x^2}}dx$

Differentiating I w.r.t. a , we have

$$\frac{dI}{da} = \frac{\pi}{2}\cdot\frac{1}{\sqrt{1+a^2}}$$

So, $I = \frac{\pi}{2}\cdot[\log(a+\sqrt{1+a^2})] + C$

Now, put $a = 0$ to get $I = 0$ and hence $C = 0$.

Hence, $I = \frac{\pi}{2}\log(a+\sqrt{1+a^2})$

5. (b) : $I_1 = \int_0^{\pi/2} \underbrace{\sin^{\sqrt{2}} x}_{\text{I}} . \underbrace{\sin x}_{\text{II}}\, dx$

Integrating by parts, we get

$\therefore$ $I_1 = \sqrt{2}I_2 - \sqrt{2}I_1 \Rightarrow \frac{I_1}{I_2} = 2 - \sqrt{2}$

6. (c): We have, $r = \frac{c^2}{\sqrt{(a\cot\theta - b\tan\theta)^2 + (a+b)^2}}$

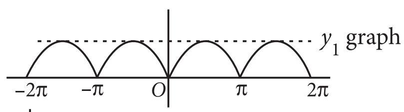
So, $r_{max.} = \frac{c^2}{(a+b)}$

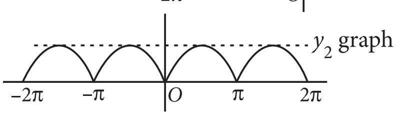
7. (c): Rewrite the fraction as

$$1000 - \frac{900a_2 + 990a_3 + 999a_1}{a_1 + a_2 + a_3 + a_4}$$

Hence, max. value = 1000

8. (a,c,d) :





9. (b,d) : We will evaluate all the four options.

For option (a), put $2x^2 = t^2$ to get $I = \dfrac{\sqrt{\pi}}{2\sqrt{2}}$

For option (b), put $x^2 = t$ to get $I = \dfrac{1}{2}$

For option (c), (d) integrating by parts

$$\int_0^{\infty} x(xe^{-x^2})dx = \frac{1}{2}\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$$

10. (a,d) : Notice that $\displaystyle\int_0^{\infty}\frac{\sin kx}{x}dx = I,$ for $k > 0$

So, given integral $\displaystyle\int_0^{\infty}\frac{\sin ax\cos bx}{x}dx$

$$= \frac{1}{2}\left[\int_0^{\infty}\frac{\sin(a+b)x}{x} + \int_0^{\infty}\frac{\sin(a-b)x}{x}dx\right]$$

$= \dfrac{1}{2}(I+I)$ for $a > b$ and $= \dfrac{1}{2}(I-I)$ for $a < b$

Hence, given integral = I for $a > b$ and zero for $a < b$.

11. (a,c,d) : The given integral can be rewritten as

$$I = \int_a^{\infty}\frac{ndx}{n^2\left(x^2+\frac{1}{n^2}\right)} = \left[\tan^{-1}nx\right]_a^{\infty} = \frac{\pi}{2} - \tan^{-1}(an)$$

Hence, $I = \pi$ if $a < 0 = \pi/2$ if $a = 0 = 0$ if $a > 0$

12. (c) : $\dfrac{f'(x)}{f(x)} = \dfrac{f'(-x)}{f(-x)} \to$ Integrating, $f(x)\cdot f(-x) = c$

$f(0) = 3 \Rightarrow c = 9 \therefore f(x)f(-x) = 9$

13. (a) : $I = \displaystyle\int_{-51}^{51}\frac{dx}{3+f(x)}$

and also, $I = \displaystyle\int_{-51}^{51}\frac{dx}{3+f(-x)}$ (replacing x by $-x$)

Adding the above two integrals and simplify

$$2I = \int_{-51}^{51}\frac{6+f(x)+f(-x)}{18+3[f(x)+f(-x)]}dx$$

i.e. $2I = \displaystyle\int_{-51}^{51}\frac{1}{3}dx$

Hence, $I = 17$

14. (a) **15. (b)**

(14-15) : $D(t) = \left|\dfrac{1}{\sqrt{(ae^t)^2+(be^{-t})^2}}\right|$

Hence, $I = \displaystyle\int_0^1 (a^2e^{2t}+b^2e^{-2t})dt = \frac{(e^2-1)}{2}\left(a^2+\frac{b^2}{e^2}\right)$

For $ab = 1$, $I = \dfrac{(e^2-1)}{2}\left[\dfrac{2}{e}+\left(a-\dfrac{1}{ae}\right)^2\right]$

So, I is minimum at $= a - \dfrac{1}{ae} = 0,$ *i.e.*, $b = \sqrt{e}$

and $I_{\text{min.}} = \dfrac{(e^2-1)}{2}\cdot\left[\dfrac{2}{e}\right] = e - \dfrac{1}{e}$

16. (2) : I = Real part of $\displaystyle\int_0^{2105\pi} e^{\cos\theta}.e^{i\sin\theta}d\theta$

$$= \text{Re}\left[\int_0^{2105\pi} e^{e^{i\theta}}d\theta\right] = \text{Re}\left[\int_0^{2105\pi} 1+e^{i\theta}+\frac{(e^{i\theta})^2}{2!}+.....\right]d\theta$$

$$= \int_0^{2105\pi}\left(1+\cos\theta+\frac{\cos 2\theta}{2!}+.....\right)d\theta$$

$$= \left[\theta+\sin\theta+\frac{\sin 2\theta}{2\times 2!}+.....\right]_0^{2105\pi} = 2105\pi$$

Hence, $\left[\dfrac{k}{1000}\right] = 2$

17. (3) : Use the identity $\dfrac{d^2x}{dy^2} = \dfrac{-d^2y/dx^2}{(dy/dx)^3}$ to differentiate again, to get the value of $k = 3$

18. (0) : Convert tan and cot into sin and cos forms and put $(\sin x)^{10} + (\cos x)^{10} = t$

19. (1) : Put $(x^2+2x)^{\frac{1}{10}} = t$ in $I = \displaystyle\int (x+1)(x^2+2x)^{\frac{1}{10}}dx$

20. (1) : Put $x = \sin^2\theta$ and use waali's formula to get

$$I = \frac{2\times(140\cdot 138\cdot 136\cdot......\cdot 2)(60\cdot 58\cdot 56\cdot......\cdot 2)}{(202)\cdot(200)\cdot.....\cdot 2}$$

$$= \frac{70!\times 30!}{101\times 100!}$$

MPP-2 CLASS XI ANSWER KEY

1.	(d)	**2.**	(a)	**3.**	(b)	**4.**	(a)	**5.**	(d)
6.	(d)	**7.**	(d)	**8.**	(a, b)	**9.**	(a, b)		
10.	(a, b, c)			**11.**	(a, b, c)				
12.	(a, b, d)			**13.**	(b, d)	**14.**	(a)	**15.**	(a)
16.	(b)	**17.**	(5)	**18.**	(1)	**19.**	(7)	**20.**	(2)

MPP-2 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

Inverse Trigonometric Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Domain of $\sin^{-1}\left\{\log_3\left(\frac{x}{3}\right)\right\}$ is

(a) $[1, 9]$ (b) $[-1, 9]$ (c) $[-9, 1]$ (d) $[-9, -1]$

2. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}........\right)$

$$+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-.......\right)=\frac{\pi}{2}$$

for $0 < |x| < \sqrt{2}$, then x equals

(a) $1/2$ (b) 1 (c) $-1/2$ (d) -1

3. Which of the following is INCORRECT?

(a) $x=\frac{1}{2}$ is the only root of the equation $3\sin^{-1}x+\pi x-\pi=0$.

(b) The numerical value of $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$ is positive.

(c) The equation $2(\sin^{-1}x)^2-5\sin^{-1}x+2=0$ has at least one solution.

(d) The equation

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)=1$$

has no solution.

4. If we consider only the principal values of the inverse trigonometric functions, then the value of

$$\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right)$$ is

(a) $\frac{\sqrt{29}}{3}$ (b) $\frac{29}{3}$ (c) $\frac{\sqrt{3}}{29}$ (d) $\frac{3}{29}$

5. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then $xy+yz+zx$ is equal to

(a) -3 (b) 0 (c) 3 (d) -1

6. If $\sin^{-1}\sqrt{(x^2+2x+1)}+\sec^{-1}\sqrt{(x^2+2x+1)}=\frac{\pi}{2}$

$x\neq 0$, then the value of $2\sec^{-1}\left(\frac{x}{2}\right)+\sin^{-1}\left(\frac{x}{2}\right)$ is equal to

(a) $-\frac{3\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

One or More than One Options Correct Type

7. Let $f(x)=e^{\cos^{-1}\sin\left(x+\frac{\pi}{3}\right)}$, then

(a) $f\left(\frac{8\pi}{9}\right)=e^{\frac{5\pi}{18}}$ (b) $f\left(\frac{8\pi}{9}\right)=e^{\frac{13\pi}{18}}$

(c) $f\left(-\frac{7\pi}{4}\right)=e^{\frac{\pi}{12}}$ (d) $f\left(-\frac{7\pi}{4}\right)=e^{\frac{11\pi}{12}}$

8. $\theta=\tan^{-1}[2\tan^2\theta]-\tan^{-1}[(1/3)\tan\theta]$ if

(a) $\tan\theta=-2$ (b) $\tan\theta=0$

(c) $\tan\theta=1$ (d) $\tan\theta=2$

9. If $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, the solution of the equation $\log_{\sin\theta}(\cos^2\theta-\sin^2\theta)=2$ is given by

(a) $\theta=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\theta=\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

(c) $\theta=n\pi$ (d) $\theta=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

10. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left\{\cos\left(-\frac{14\pi}{5}\right)\right\}\right]$, is

(a) $\cos\left(-\frac{7\pi}{5}\right)$ (b) $\sin\left(\frac{\pi}{10}\right)$

(c) $\cos\left(\frac{2\pi}{5}\right)$ (d) $-\cos\left(\frac{3\pi}{5}\right)$

11. If $\cos^{-1}x = \tan^{-1}x$, then

(a) $x^2 = (\sqrt{5}-1)/2$ (b) $x^2 = (\sqrt{5}+1)/2$

(c) $\sin(\cos^{-1}x) = (\sqrt{5}-1)/2$

(d) $\tan(\cos^{-1}x) = (\sqrt{5}-1)/2$

12. The value of $\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}$ $+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 0

13. Identify the pair(s) of functions which are identical.

(a) $y = \tan(\cos^{-1}x);\ y = \frac{\sqrt{1-x^2}}{x}$

(b) $y = \tan(\cot^{-1}x);\ y = \frac{1}{x}$

(c) $y = \sin(\text{arc}\tan x);\ y = \frac{x}{\sqrt{1+x^2}}$

(d) $y = \cos(\text{arc}\tan x);\ y = \sin(\text{arc}\cot x)$

Comprehension Type

We know that corresponding to every bijection $g: B \to A$ defined by $g(y) = x$ if and only if $f(x) = y$

The function $g: B \to A$ is called the inverse of function $f: A \to B$ and is denoted by f^{-1}.

Thus, we have $f(x) = y \Rightarrow f^{-1}(y) = x$.

We have also learnt that

$(f^{-1} of)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$, for all $x \in A$

and $(fof^{-1})(y) = f(f^{-1}(y)) = f(x) = y$, for all $y \in B$.

We know that trigonometric functions are periodic functions and hence, in general all trigonometric functions are not bijectives. Consequently, their inverse do not exist. However, if we restrict their domains and co-domains, they can be made bijections and we can obtain their inverse.

14. The value of x, if $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, is

(a) $[-1, 1]$ (b) $\left[-\frac{1}{\sqrt{2}}, 1\right]$

(c) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (d) none of these

15. The value of, $\sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}]$, where $x \in \left(\frac{\pi}{2}, \pi\right)$, is

(a) $\frac{\pi}{2}$ (b) $-\pi$ (c) π (d) $-\frac{\pi}{2}$

Matrix Match Type

16. Match the following.

	Column I	Column II	
(P)	If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then $x =$	(1)	π
(Q)	If $(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = \frac{5\pi^2}{8}$, then $x =$	(2)	$\frac{\pi}{2}$
(R)	If $x + y + z = xyz$, then $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z =$	(3)	$\frac{-1}{\sqrt{2}}$
(S)	If $xy + yz + zx = 1$, then $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z =$	(4)	-1

	P	Q	R	S
(a)	1	3	2	4
(b)	4	3	1	2
(c)	4	3	1	2
(d)	1	2	4	3

Integer Answer Type

17. If $S = \sum_{r=1}^{50}\tan^{-1}\left(\frac{2r}{2+r^2+r^4}\right)$, then the value of $\frac{2550}{2552}$ cot S must be

18. The number of points (x, y) inside the circle $x^2 + y^2 = 4$ satisfying the equation $\tan^4 x + \cot^4 x + 1 = 3\sin^2 y - 1$ is

19. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, $\cos^{-1}x - \cos^{-1}y = -\frac{\pi}{3}$, then the number of values of (x, y) is

20. If $\theta = \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$, then the value of $\cot^2\theta$ is

◈◈

Keys are published in this issue. Search now! ☺

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 164

JEE MAIN

1. The number of points at which the function $y = f(x) = \left|1 - \left|1 - |x|\right|\right|$ is not differentiable is
(a) 2 (b) 3
(c) 4 (d) 5

2. Let A and B be two sets. The set A has 2016 more subsets than B. If $A \cap B$ has 3 members, then the number of members of $A \cup B$ is
(a) 10 (b) 11 (c) 12 (d) 13

3. If $\frac{1}{4} + \frac{1.2}{4.5} + \frac{1.2.3}{4.5.6} + \ldots$ to 20 terms $= \frac{m}{n}$, reduced fraction, then $n - 2m =$
(a) 1 (b) 2 (c) 3 (d) 4

4. Ten boys and two girls are to be seated in a row such that there are atleast 3 boys between the girls. The number of ways this can be done is $\lambda \cdot 12!$, where $\lambda =$
(a) 2/3 (b) 4/11
(c) 5/11 (d) 6/11

5. AB is a chord of the circle $x^2 + y^2 = 25$. The tangents at A and B meet at C. If $M(2, 3)$ is the mid-point of AB, then the area of the kite $OACB$ is
(a) $50\sqrt{\frac{13}{3}}$ (b) $50\sqrt{\frac{3}{13}}$
(c) $50\sqrt{3}$ (d) $\frac{50}{\sqrt{13}}$

JEE ADVANCED

6. In triangle ABC, if $a = x^2 + x + 1$, $b = x^2 - 1$, $c = 2x + 1$ and $C = \pi/6$, then $x =$
(a) $-(2+\sqrt{3})$ (b) $1+\sqrt{3}$
(c) $2+\sqrt{3}$ (d) $4\sqrt{3}$

COMPREHENSION

In a triangle ABC, $r_1 = 5$, $r_2 = 3$ and $C = \frac{\pi}{2}$.

7. $a + b =$
(a) $\sqrt{31}$ (b) $\sqrt{35}$ (c) $\frac{\sqrt{35}}{2}$ (d) $2\sqrt{31}$

8. $r =$
(a) $\sqrt{35} - 1$ (b) $\sqrt{35} - 2$
(c) $\sqrt{31} - 4$ (d) $\sqrt{31} - 2$

INTEGER MATCH

9. If the sum $\frac{1}{7} + \frac{1\cdot3}{7\cdot9} + \frac{1\cdot3\cdot5}{7\cdot9\cdot11} + \ldots$ to 20 terms is $\frac{m}{n}$, reduced fraction, then $n - 4m =$

MATRIX MATCH

10. The curve $f(x, y) = 0$ lies in the first quadrant. The tangent at a point on it meets the positive x and y axes at A and B and O is the origin

	List-I		List-II
P.	$f(x, y) = 4xy - 1$	1.	$AB = 1$
Q.	$f(x, y) = x^2 + y^2 - 1$	2.	$OA + OB = 1$
R.	$f(x, y) = \sqrt{x} + \sqrt{y} - 1$	3.	$OA \cdot OB = 1$
S.	$f(x, y) = x^{2/3} + y^{2/3} - 1$	4.	$\frac{1}{OA^2} + \frac{1}{OB^2} = 1$

	P	Q	R	S
(a)	2	1	3	4
(b)	1	2	4	3
(c)	4	3	2	1
(d)	3	4	2	1

See Solution set of Maths Musing 163 on page no. 84

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main and Advanced) and other PETs. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for competitions. In every issue of MT, challenging problems are offered with detailed solutions. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. The line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + 1 = z$ and $x + 1 = 2y = 2z$. The coordinates of each of the points of intersection are given by

(a) (3, 3, 3), (1, 1, 1) (b) (3, 2, 3), (1, 1, 1)
(c) (3, 2, 3), (1, 1, 2) (d) (2, 3, 3), (2, 1, 1)

2. Total number of common tangents of $y^2 = 4ax$ and $xy = c^2$, is equal to

(a) 1 (b) 2 (c) 3 (d) 4

3. Let $f : R \to R$ be a function defined by $f(x) = [x]^2 + [x + 1] - 3$ {where [·] denotes greatest integer function}, then $f(x)$ is

(a) many-one into (b) many-one onto
(c) one-one into (d) one-one onto

4. If $2^{3n} - \alpha n - \beta$ is divisible by 49, then (α, β) is, $n \in N$

(a) (–7, – 1) (b) (7, 1)
(c) (49, 1) (d) (49, 7)

5. The length of projection of the line segment joining the points (1, –1, 0) and (–1, 0, 1) to the plane $2x + y + 6z = 1$ is equal to

(a) $\sqrt{\dfrac{255}{41}}$ units (b) $\sqrt{\dfrac{237}{41}}$ units
(c) $\sqrt{\dfrac{137}{41}}$ units (d) $\sqrt{\dfrac{155}{41}}$ units

6. $\displaystyle\sum_{r=1}^{15} \frac{r2^r}{(r+2)!}$ is equal to

(a) $\dfrac{17!-2^{16}}{17!}$ (b) $\dfrac{18!-2^{17}}{18!}$
(c) $\dfrac{16!-2^{15}}{16!}$ (d) $\dfrac{15!-2^{14}}{15!}$

7. There are n locks and n matching keys. If all the locks and keys are to be perfectly matched, then maximum number of trials is equal to

(a) ${}^nC_2$ (b) ${}^{n-1}C_2$
(c) ${}^{n+1}C_2$ (d) none of these

8. Maximum length of the chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\,(a > b)$ such that eccentric angles of its extremities differ by $\dfrac{\pi}{2}$ is

(a) $a\sqrt{2}$ (b) $b\sqrt{2}$
(c) $ab\sqrt{2}$ (d) none of these

9. The number of ordered pairs (m, n), ($m, n \in$ {1, 2, ..., 20}) such that $3^m + 7^n$ is a multiple of 10, is

(a) 100 (b) 200
(c) $4! \times 4!$ (d) none of these

10. If the length of the tangent drawn from a variable point to one given circle is k ($\neq 1$) times the length of the tangent from it to another circle, the locus of the variable point is

(a) an ellipse (b) a parabola
(c) a circle (d) a hyperbola

SOLUTIONS

1. (b) : The first line $x = y + 1 = z = \mu$ gives general point as $(\mu, \mu - 1, \mu)$.

Second line $x + 1 = 2y = 2z = \lambda$ gives general point as $(\lambda - 1, \lambda/2, \lambda/2)$. The ratios of direction ratios of line

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joining the point of intersection and direction cosines proportional to 2, 1, 2 are same

i.e., $\frac{\mu-\lambda+1}{2}=\frac{\mu-1-\frac{\lambda}{2}}{1}=\frac{\mu-\frac{\lambda}{2}}{2}$

$\Rightarrow\ \mu-\lambda+1=2\mu-2-\lambda=\mu-\frac{\lambda}{2}$

Solving, we get $\mu = 3, \lambda = 2$

$\therefore$ Coordinates are (3, 2, 3), (1, 1, 1)

2. (a) : Any tangent of $xy = c^2$ is $\frac{x}{t}+yt=2c$

i.e., $y=-\frac{x}{t^2}+\frac{2c}{t}$

Comparing it with $y = mx+\frac{a}{m}$, we get

$$m=-\frac{1}{t^2}, m=\frac{at}{2c} \Rightarrow \frac{at}{2c}=-\frac{1}{t^2} \Rightarrow t=-\left(\frac{2c}{a}\right)^{1/3}$$

So, there is only one common tangent.

3. (a) : $f(x) = [x]^2 + [x] - 2 = ([x] + 2)([x] - 1)$

$f(x) = 0$ for $x \in [1, 2)$ and $x \in [-2, -1)$

So $f(x)$ is many one. As $f(x)$ will take only integral values, so it is into.

4. (b) : $8^n - \alpha n - \beta = (1 + 7)^n - \alpha n - \beta$

$= 1 + 7n + {}^nC_2\, 7^2 + {}^nC_3 7^3 + ...+ {}^nC_n 7^n - \alpha n - \beta$

$= (1 - \beta) + (7 - \alpha)n + 49\,({}^nC_2 + 7\,{}^nC_3 +......)$

$\therefore$ If $8^n - \alpha n - \beta$ is divisible by 49, then

$1 - \beta = 0, 7 - \alpha = 0 \Rightarrow \beta = 1, \alpha = 7.$

5. (b) : $A \equiv (1, -1, 0), B \equiv (-1, 0, 1)$

Direction ratios of segment AB are 2, –1, –1.

If θ be the acute angle between segment AB and normal to plane, then

$$\cos\theta=\frac{|2\times2-1\times1-1\times6|}{\sqrt{4+1+36}\,\sqrt{4+1+1}}=\frac{3}{\sqrt{246}}$$

Length of projection = $AB\sin\theta$

$$=\sqrt{6}\sqrt{1-\frac{9}{246}}=\sqrt{\frac{237}{41}}\text{ units.}$$

6. (a) : $\frac{r2^r}{(r+2)!}=\frac{(r+2-2)2^r}{(r+2)!}$

$$=\frac{2^r}{(r+1)!}-\frac{2^{r+1}}{(r+2)!}=-\left(\frac{2^{r+1}}{(r+2)!}-\frac{2^r}{(r+1)!}\right)$$

$$\therefore\ \sum_{r=1}^{15}\frac{r2^r}{(r+2)!}=-\left(\frac{2^{16}}{17!}-\frac{2}{2!}\right)=1-\frac{2^{16}}{17!}.$$

7. (c) : For the first key, maximum number of trials needed is n. For second key, it will be $(n - 1)$ and so on.

In general, for r^{th} key, maximum number of trials needed is $(n - r + 1)$.

Then total number of trials needed

$$= n + (n - 1) + (n - 2) + ... + 1=\frac{n(n+1)}{2}={}^{n+1}C_2.$$

8. (a) : Let the extremities of the chord be $P_1 \equiv (a\cos\theta, b\sin\theta)$ and $P_2 \equiv (-a\sin\theta, b\cos\theta)$

$$(P_1P_2)^2=a^2(\cos\theta+\sin\theta)^2+b^2(\sin\theta-\cos\theta)^2$$
$$=a^2+b^2+(a^2-b^2)\sin2\theta\le a^2+b^2+a^2-b^2$$

$\Rightarrow\ (P_1P_2)^2\le 2a^2 \Rightarrow\ P_1P_2\le a\sqrt{2}$

9. (a) : The last digit of any power of 3 can be 3, 9, 7, 1. Similarly last digit of any power of 7 can be 7, 9, 3, 1

$\Rightarrow$ Total number of ways $= 5\times5+5\times5+5\times5+5\times5 = 100.$

10. (c) : $\sqrt{S_1}=k\sqrt{S_2} \Rightarrow\ S_1=k^2S_2$

So locus is $x^2 + y^2 + 2g_1x + 2f_1y + c_1$
$= k^2 (x^2 + y^2 + 2g_2x + 2f_2y + c_2)$

$\Rightarrow\ (x^2 + y^2)(1 - k^2) + 2x(g_1 - k^2g_2) + 2y(f_1 - k^2 f_2) + c_1 - k^2c_2 = 0$ which is a circle.

WB JEE MOCK TEST PAPER

Series-1

The entire syllabus of Mathematics of WB-JEE is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow:

Unit No	Topic	Syllabus In Details
UNIT NO. 1	**Sets, Relations & Functions**	Sets and their representation; Union, intersection and compliment of sets and their algebraic properties; Power set. Ordered pairs, Cartesian product of sets. Definition of relation. Domain and range of a relation. Types of relations, equivalence relations. Function as a special kind of relation from one set to another. Domain, co-domain and range of a function. One-one, into and onto functions, Composition of functions. Real valued function of a real variable, domain and range of these functions, constant, identity, polynomial, rational, modulus, signum and greatest integer functions with their graphs, sum, difference, product and quotients of functions.
	Trigonometry	Measurement of Trigonometric Angles, Definition of Trigonometric functions, Associated angles
	Co-ordinate Geometry-2D	Cartesian coordinates, distance between two points, section formulae, shift of origin, Locus problems.

Time : 1 hr 15 min **Full marks : 50**

CATEGORY-I

For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.

1. Let $X_n = \left\{z = x + iy : |z|^2 \le \frac{1}{n}\right\}$ for all integers $n \ge 1$. Then $\bigcap_{n=1}^{\infty} X_n$ is

(a) a singleton set (b) not a finite set

(c) an empty set

(d) a finite set with more than one element.

2. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is

(a) $(-\infty, 0)$ (b) $(-\infty, \infty) - \{0\}$

(c) $(-\infty, \infty)$ (d) $(0, \infty)$

3. The degree measure of radian measure of angle $(-3)^c$ is

(a) $-170°49'5''$ (b) $-171°49'5''$

(c) $-171°50'$ (d) $-171°51'$

4. If the sum of distances from a point P on two mutually perpendicular straight lines is 1 unit, then the locus of P is

(a) a parabola (b) a circle

(c) an ellipse (d) a straight line

5. The line joining $A(b\cos\alpha, b\sin\alpha)$ and $B(a\cos\beta, a\sin\beta)$, where $a \ne b$ is produced to the point $M(x, y)$ so that $AM : BM = b : a$. Then $x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2}$ is equal to

(a) 0 (b) 1 (c) -1 (d) $a^2 + b^2$

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6. If C is a point on the line segment joining $A(-3, 4)$ and $B(2, 1)$ such that $AC = 2BC$, then the co-ordinates of C is
(a) $\left(\frac{1}{3}, 2\right)$ (b) $\left(2, \frac{1}{3}\right)$
(c) $(2, 7)$ (d) $(7, 2)$

7. If a and b are positive, then the minimum value of $\frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x}$ is
(a) $a + b$ (b) $a^2 + b^2$
(c) $(a + b)^2$ (d) $(a - b)^2$

8. The smallest value of $5\cos\theta + 12$ is
(a) 5 (b) 12 (c) 7 (d) 17

9. The range of $y = \frac{|\sin x|}{1+|\sin x|}$ is
(a) $0 < y < 1$ (b) $0 \le y \le 1$
(c) $0 \le y < 1$ (d) none of these

10. If $f\{g(x)\} = 1 + \sin x \cos x$ and $2f(x) = 1 + x^2$, then $g(x)$ equals
(a) $\sin x + \cos x$ (b) $|\sin x + \cos x|$
(c) $(\sin x + \cos x)^2$ (d) $|\cos x - \sin x|$

11. For any two sets A and B, $A - (A - B)$ equals
(a) B (b) $A - B$
(c) $A \cap B$ (d) $A^c \cap B^c$

12. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from the set A to set B is
(a) 2^9 (b) 9^2 (c) 3^2 (d) $2^9 - 1$

13. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$; then which of the following relation is a function from A to B?
(a) $\{(1, 2), (2, 3), (3, 4), (2, 2)\}$
(b) $\{(1, 2), (2, 3), (1, 3)\}$
(c) $\{(1, 3), (2, 3), (3, 3)\}$
(d) $\{(1, 1), (2, 3), (3, 4)\}$

14. The identity mapping $I_c : S \to S$ defined as $I_s(x) = x$ for $x \in S$. Suppose $f : A \to B$ is a bijection, then which of the following is true?
(a) $f^{-1} o\, f \ne I_A$ but $fof^{-1} = I_B$
(b) $f^{-1} o\, f = I_A$ but $fof^{-1} = I_B$
(c) $f^{-1} o\, f = I_A$ but $fof^{-1} \ne I_B$
(d) $f^{-1} o\, f \ne I_A$ but $fof^{-1} \ne I_B$.

15. Let $R = \{(3,3), (6,6), (9,9), (12,12), (6,12), (3,9), (3,12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$. The relation is
(a) reflexive and transitive only
(b) reflexive only
(c) an equivalence relation
(d) reflexive and symmetric only

16. If $2(\sin^6 x + \cos^6 x) + \lambda(\sin^4 x + \cos^4 x) = -1$, then the value of λ is
(a) 0 (b) -1 (c) -2 (d) -3

17. If C is the reflection of $A(2, 4)$ in x-axis and B is the reflection of C in y-axis, then $|AB|$ is
(a) 20 (b) $2\sqrt{5}$ (c) $4\sqrt{5}$ (d) 4

18. The co-ordinates of a moving point P are $(2t^2 + 4, 4t + 6)$. Then its locus will be a
(a) circle (b) straight line
(c) parabola (d) ellipse

19. The cartesian co-ordinates of a point are $(1, -1)$, its polar co-ordinates are
(a) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (b) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$
(c) $\left(\sqrt{2}, \frac{5\pi}{4}\right)$ (d) $\left(\sqrt{2}, \frac{7\pi}{4}\right)$

20. There are 100 students in a class. In an examination, 50 of them failed in Mathematics, 45 failed in Physics, 40 failed in biology, 32 failed in exactly two of the three subjects. Only one student passed in all the subjects. Then the number of students failing in all three subjects
(a) 12 (b) 4
(c) 2 (d) can't determined

21. Let $f(x) = x\left(\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1}\right), x > 1$, then
(a) $f(x) \le 1$ (b) $1 < f(x) \le 2$
(c) $2 < f(x) \le 3$ (d) $f(x) > 3$

22. Let $f : R \to R$ such that $f(x) = x^3 + 5x + 1$, then
(a) f is neither one-one nor onto
(b) f is one-one but not onto
(c) f is onto but not one-one
(d) f is one-one and onto

23. Let $f : (2, 4) \to (1, 3)$, where $f(x) = x - \left[\frac{x}{2}\right]$ (where $[\cdot]$ denotes the greatest integer function), then $f^{-1}(x)$ is
(a) not defined (b) $x - 1$
(c) $x + 1$ (d) none of these

24. If $f(x+f(y)) = f(x) + y \ \forall \ x, y \in R$ and $f(0) = 1$, then $f(7) =$

(a) 1 (b) 0 (c) –1 (d) 7

25. The domain of $y = \sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$ is

(a) $[1, 6)$ (b) $\left[1, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 6\right]$

(c) $\left[1, \pi\right] \cup \left[\frac{7\pi}{4}, 6\right]$ (d) none of these

26. If $[2\sin x] + [\cos x] = -3$ then the range of the function $f(x) = \sin x + \sqrt{3}\cos x$ in $[0, 2\pi]$ is (where $[\cdot]$ denotes the greatest integer function)

(a) $[-2, -1)$ (b) $(-2, -1)$

(c) $\left(-1, -\frac{1}{2}\right)$ (d) none of these

27. The smallest positive values of x and y which satisfy $\tan(x - y) = 1$; $\sec(x + y) = \frac{2}{\sqrt{3}}$ are

(a) $x = \frac{37\pi}{24}, y = \frac{7\pi}{24}$ (b) $x = \frac{15\pi}{24}, y = \frac{19\pi}{24}$

(c) $x = \frac{\pi}{3}, y = \frac{7\pi}{12}$ (d) $x = \frac{\pi}{4}, y = \frac{\pi}{2}$

28. Let α be a solution of $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$ in $\left(0, \frac{\pi}{4}\right)$. If the shadow of a pole is $\frac{1}{\sqrt{3}}$ of its height, then the altitude of the sun is

(a) α (b) $\frac{\alpha}{2}$ (c) 2α (d) $\frac{\alpha}{3}$

29. If $\sin x = \cos^2 x$, then $\cos^2 x(1 + \cos^2 x) =$

(a) 0 (b) 1 (c) 2 (d) none of these

30. P, Q, R and S are the mid-points of sides of a quadrilateral $ABCD$ in that order. If the co-ordinates of the points P, Q, R and S are (0, 1), (3, 5), (4, 6) and (a, b) respectively, then

(a) $a = 1, b = 3$ (b) $a = 2, b = 3$

(c) $a = 1, b = 2$ (d) $a = 2, b = 4$

CATEGORY-II

Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.

31. Let $f : R \to R$ be a function defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$ then

(a) f is one-one and onto

(b) f is not one-one but onto

(c) f is not one-one but onto

(d) f is neither one-one nor onto

32. Let $f(x) = \log_e x + \log_x e$, then the domain of the function $\frac{1}{\sqrt{|f(x)| - f(x)}}$ is

(a) $(0, 1)$ (b) $(1, \infty)$

(c) $(1, e)$ (d) $(-\infty, \infty)$

33. Let $f : R \to R$ defined by $f(x) = x^3 + x^2 + 100x + 5\sin x$, then f is

(a) many one onto (b) many-one into

(c) one-one onto (d) one-one into

34. The relation R in $N \times N$ such that $(a, b)R\,(c, d)$ iff $a + d = b + c$ is

(a) reflexive but not symmetric

(b) reflexive and transitive but not symmetric

(c) an equivalence relation

(d) none of these

35. If $\tan 15° = 2 - \sqrt{3}$, then $2\tan 1095° + \cot 975° + \tan(-195°) =$

(a) $2 + \sqrt{3}$ (b) $4 + 2\sqrt{3}$

(c) $4 - 2\sqrt{3}$ (d) $2 - \sqrt{3}$

CATEGORY-III

In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates mark one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without making any incorrect, formula below will be used to allot marks. 2 × (no. of correct response/total no. of correct options)

36. Point R divides line joining $A(-5, 1)$ and $B(3, 5)$ in the ratio $\lambda : 1$. The co-ordinates of P and Q are (1, 5) and (7, 2) respectively. If the area of the triangle PQR be 2 sq. units, then the value of λ is

(a) $\frac{19}{5}$ (b) $\frac{31}{9}$ (c) 23 (d) 19

37. If $\tan\theta = -\frac{1}{\sqrt{5}}$, then the value of $\cos\theta =$

(a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{1}{\sqrt{6}}$ (c) $-\frac{\sqrt{5}}{\sqrt{6}}$ (d) $\frac{1}{2}$

38. If $f(x) = 27x^3 + \frac{1}{x^3}$ and α, β are the roots of $3x + \frac{1}{x} = 2$, then

(a) $f(\alpha) = f(\beta)$ (b) $f(\alpha) = 10$
(c) $f(\beta) = -10$ (d) none of these

39. Let $f(x) = [x]^2 + [x + 1] - 3$, where $[x]$ = the greatest integer $\le x$. Then
(a) $f(x)$ is a many-one and into function
(b) $f(x) = 0$ for infinite number of values of x
(c) $f(x) = 0$ for only two real values of x
(d) none of these

40. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right-angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the values of 'k' is
(a) 0 (b) –1 (c) 2 (d) 3

SOLUTIONS

1. (a) : $X_n = \left\{z = x + iy : |z|^2 \le \frac{1}{n}\right\}, \forall\, n \ge 1$

Now, $|z|^2 \le \frac{1}{n} \Rightarrow x^2 + y^2 \le \frac{1}{n}$

$\Rightarrow x^2 + y^2 \le 0$, when $n \to \infty$

The above inequation is true only when $x = 0$, $y = 0$

So $\bigcap\limits_{n=1}^{\infty} X_n$ will be a point circle which is the origin.

2. (a) : Given, $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

Clearly, $|x| - x > 0$

$\Rightarrow |x| > x \Rightarrow x$ is negative

$\therefore$ The domain of the function $f(x)$ is $(-\infty, 0)$

3. (b) : $(-3)^c = \left(\dfrac{180}{\pi} \times -3\right)^\circ = \left(\dfrac{180}{22} \times 7 \times (-3)\right)^\circ$

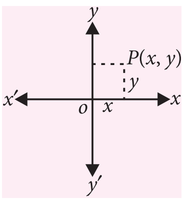
$= \left(-171\dfrac{9}{11}\right)^\circ = -171^\circ\left(\dfrac{9}{11} \times 60\right)' = -171^\circ 49'\left(\dfrac{1}{11} \times 60\right)''$

$= -171^\circ 49'\left(5\dfrac{5}{11}\right)'' \cong -171^\circ 49' 5''$

4. (d) : Let the point P be (x, y)

$\therefore \; x + y = 1$

Thus the locus of P is a straight line.



5. (a) : Given that $AM : BM = b : a$

$\Rightarrow \dfrac{AM}{BM} = \dfrac{b}{a} \Rightarrow \dfrac{AB + BM}{BM} = \dfrac{b}{a}$

$\Rightarrow 1 + \dfrac{AB}{BM} = \dfrac{b}{a} \Rightarrow \dfrac{AB}{BM} = \dfrac{b - a}{a}$

$\therefore$ B divides AM in the ratio $b - a : a$

$\therefore \; a\cos\beta = \dfrac{(b - a)x + ab\cos\alpha}{b}$

$\Rightarrow (b - a)x + ab\cos\alpha = ab\cos\beta$

$\Rightarrow (b - a)x = ab(\cos\beta - \cos\alpha)$ ...(i)

Also, $a\sin\beta = \dfrac{(b - a)y + ab\sin\alpha}{b}$

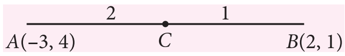
$\Rightarrow (b - a)y = ab(\sin\beta - \sin\alpha)$ ...(ii)

(i) ÷ (ii) given

$$\frac{x}{y} = \frac{\cos\beta - \cos\alpha}{\sin\beta - \sin\alpha} = \frac{2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\beta-\alpha}{2}}$$

$$\Rightarrow \frac{x}{y} = \frac{-\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \Rightarrow x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0$$

6. (a) : Given that $AC = 2BC$



$\Rightarrow AC : BC = 2 : 1$

$\therefore \; C \equiv \left(\dfrac{2\times 2 + 1(-3)}{2 + 1}, \dfrac{2\times 1 + 1\times 4}{2 + 1}\right) \equiv \left(\dfrac{1}{3}, 2\right)$

7. (c) : $\dfrac{a^2}{\cos^2 x} + \dfrac{b^2}{\sin^2 x} = a^2\sec^2 x + b^2\mathrm{cosec}^2 x$

$= a^2(1 + \tan^2 x) + b^2(1 + \cot^2 x)$

$= a^2 + b^2 + (a\tan x)^2 + (b\cot x)^2$

$= a^2 + b^2 + (a\tan x - b\cot x)^2 + 2ab\tan x\cot x$

$= a^2 + b^2 + 2ab + (a\tan x - b\cot x)^2$

$= (a + b)^2 + (a\tan x - b\cot x)^2$

$\therefore \; \dfrac{a^2}{\cos^2 x} + \dfrac{b^2}{\sin^2 x} \ge (a + b)^2$

8. (c) : We know that $-1 \le \cos\theta \le 1$

$\Rightarrow -5 \le 5\cos\theta \le 5 \Rightarrow -5 + 12 \le 5\cos\theta + 12 \le 5 + 12$

$\Rightarrow 7 \le 5\cos\theta + 12 \le 17$

$\therefore$ The minimum value of the given expression is 7.

9. (d) : Here, $y = \dfrac{|\sin x|}{1 + |\sin x|}$

$\Rightarrow y + y|\sin x| = |\sin x| \Rightarrow (1 - y)|\sin x| = y$

$\Rightarrow |\sin x| = \dfrac{y}{1 - y}$

$\because \; 0 \le |\sin x| \le 1 \Rightarrow 0 \le \dfrac{y}{1 - y} \le 1 \Rightarrow 0 \le y \le 1 - y$

$\Rightarrow 0 \le 2y \le 1 \Rightarrow 0 \le y \le \dfrac{1}{2}$

10. (b) : $\because\ 2f(x)=1+x^2 \Rightarrow 2f\{g(x)\} = 1 + \{(g(x)\}^2$

$\because\ f(g(x))=1+\sin x\cos x$

$\therefore\ 1 + \{g(x)\}^2 = 2\ (1 + \sin x\cos x)$

$\Rightarrow\ \{g(x)\}^2 = 1 + 2\sin x\cos x = (\cos x + \sin x)^2$

$\Rightarrow\ g(x)=|\sin x+\cos x|$

11. (c) : $A - (A - B) = A \cap (A - B)^c = A \cap (A \cap B^c)^c$
$= A \cap (A^c \cup B) = (A \cap A^c) \cup (A \cap B)$ [distributive law]
$= \phi \cup (A \cap B) = A \cap B$

12. (a) : Here, $A = \{2, 4, 6\}$ and $B = \{2, 3, 5\}$

$\therefore\ n\ (A \times B) = 3 \times 3 = 9$

$\therefore$ Number of relations from set A to set $B = 2^9$

13. (c) : Since, each element of A has only one image in Set B

$\therefore$ Option (c) is a function.

14. (b)

15. (a) : Since $(3, 3), (6, 6), (9, 9), (12, 12) \in R$

Hence R is reflexive

$\because\ (3, 6), (6, 12)$ and $(3, 12) \in R$. Therefore R is transitive.

$\because\ (3, 6) \in R$ but $(6, 3) \notin R$, hence R is not symmetric.

16. (d) : $2(\sin^6 x + \cos^6 x) + \lambda\ (\sin^4 x + \cos^4 x) = -1$

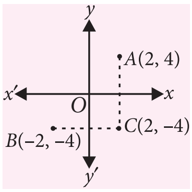
$\Rightarrow\ 2[(\sin^2 x)^3 + (\cos^2 x)^3] + \lambda\ [(\sin^2 x)^2 + (\cos^2 x)^2] = -1$

$\Rightarrow\ 2[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)]$
$+ \lambda[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x] = -1$

$\Rightarrow\ 2 - 6\sin^2 x\cos^2 x + \lambda\ (1 - 2\sin^2 x\cos^2 x) = -1$

Clearly $\lambda = -3$

17. (c) : Clearly $C(2, -4)$ and $B(-2, -4)$.



$\therefore\ AB=\sqrt{(2+2)^2+(4+4)^2}$
$=\sqrt{16+64}=\sqrt{80}=4\sqrt{5}$

18. (c) : Let $x = 2t^2 + 4$ and $y = 4t + 6$

$\therefore\ x=2\left(\frac{y-6}{4}\right)^2+4=\frac{2(y-6)^2}{16}+4=\frac{(y-6)^2}{8}+4$

$\Rightarrow\ 8x = (y - 6)^2 + 32 \Rightarrow (y - 6)^2 = 8x - 32$

$\Rightarrow\ (y - 6)^2 = 8(x - 4)$

$\therefore$ The locus of the moving point is a parabola.

19. (d) : The point $(1, -1)$ lies in 4^{th} quadrant.

Here $x = 1, y = -1$ $\quad\therefore\ r=\sqrt{(1)^2+(-1)^2}=\sqrt{2}$

$\tan\theta = -1 = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow\ \tan\theta = \tan\left(2\pi-\frac{\pi}{4}\right)=\tan\frac{7\pi}{4} \Rightarrow \theta=\frac{7\pi}{4}$

$\therefore$ The required polar co-ordinates are $\left(\sqrt{2}, \frac{7\pi}{4}\right)$

20. (c) : Let A, B and C be the sets of students who failed in Mathematics, Physics and Biology respectively.

According to question,

$n(A \cup B \cup C) = 100 - 1 = 99$

$n(A) = 50, n(B) = 45, n(C) = 40$

and $n(A \cap B) + n(B \cap C) + n(A \cap C)$
$- 3n(A \cap B \cap C) = 32$

Now, using formula,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$\therefore\ 99 = 50 + 45 + 40 - 3n(A \cap B \cap C) - 32$
$+ n(A \cap B \cap C)$

$\Rightarrow\ 99 = 135 - 32 - 2n(A \cap B \cap C)$

$\Rightarrow\ 2n(A \cap B \cap C) = 103 - 99 = 4$

$\therefore\ n(A \cap B \cap C) = 2$

21. (d) : Here $f(x)=x\left(\frac{1}{x-1}+\frac{1}{x}+\frac{1}{x+1}\right), x>1$

$=x\left(\frac{x+1+x-1}{x^2-1}+\frac{1}{x}\right)$

$=\frac{2x^2}{x^2-1}+1=\frac{3x^2-1}{x^2-1}=\frac{3(x^2-1)+2}{x^2-1}=3+\frac{2}{x^2-1}$

$\because\ x>1\ \therefore\ 3+\frac{2}{x^2-1}>3$ or $f(x)>3$

22. (d) : Let x_1 and x_2 are two real numbers.

Now $f(x_1) = f(x_2) \Rightarrow\ x_1^3+5x_1+1=x_2^3+5x_2+1$

$\Rightarrow\ x_1^3-x_2^3+5(x_1-x_2)=0$

$\Rightarrow\ (x_1-x_2)(x_1^2+x_1x_2+x_2^2+5)=0$

$\Rightarrow\ x_1 - x_2 = 0\quad [\because x_1, x_2 \in R$ then $x_1^2+x_1x_2+x_2^2+5>0]$

$\Rightarrow\ x_1 = x_2$

$\therefore\ f$ is one-to-one mapping.

Again, let $y = x^3 + 5x + 1 \Rightarrow x^3 + 5x + 1 - y = 0$

This is a cubic equation is x

$\because$ every odd degree equation must have a real root.

$\therefore$ This equation must have a real root

$\therefore$ For each element y of co-domain R must have a value of x for which $f(x) = y$

$\therefore\ f$ is onto mapping.

23. (c) : $f(x)=x-\left[\frac{x}{2}\right], f:(2,4)\to(1,3)$

$2<x<4\Rightarrow 1<\frac{x}{2}<2\Rightarrow\left[\frac{x}{2}\right]=1$

$\Rightarrow\ f(x) = x - 1 \Rightarrow f^{-1}(x) = x + 1$

24. (a) : Given that
$f[(x) + f(y)] = f(x) + y \ \forall\ x, y \in R$
Putting $y = 0$, we get
$f(x + f(0)) = f(x) + 0 \Rightarrow f(x + 1) = f(x)$ $[\because f(0) = 1]$
Putting $x = 0, 1, ..., 6$, we get
$f(1) = f(0) = 1,$
$f(2) = f(1) = 1,$
$f(3) = f(2) = 1,$
$\vdots$
$f(7) = f(6) = 1$

25. (b) : $y = \sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$

$$\Rightarrow \ y = \sqrt{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)} + \sqrt{7x - x^2 - 6}$$

For the function to be defined

$$\sin\left(x + \frac{\pi}{4}\right) \ge 0 \ \Rightarrow \ 2n\pi - \frac{\pi}{4} \le x \le 2n\pi + \frac{3\pi}{4} \quad ...(i)$$

where $n = 0, \pm1, \pm2, ...$
Also $7x - x^2 - 6 \ge 0$
$\Rightarrow \ x^2 - 7x + 6 \le 0 \ \Rightarrow (x - 1)(x - 6) \le 0$
$\Rightarrow 1 \le x \le 6$ ...(ii)

From (i) & (ii), we get, $x \in \left[1, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 6\right]$

26. (b) : $[2\sin x] + [\cos x] = -3$ only if $[2\sin x] = -2$ and $[\cos x] = -1$
$\therefore \ -2 \le 2\sin x < -1$ and $-1 \le \cos x < 0$

$$\Rightarrow \ -1 \le \sin x < -\frac{1}{2} \text{ and } -1 \le \cos x < 0$$

$$\Rightarrow \ \frac{7\pi}{6} < x < \frac{11\pi}{6} \text{ and } \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Common values of x are given by $\frac{7\pi}{6} < x < \frac{3\pi}{2}$
For these values of x,

$$\sin x + \sqrt{3}\cos x = 2\sin\left(\frac{\pi}{3} + x\right) \text{ lies between } -2 \text{ and } -1$$

$\therefore$ Range of $f(x)$ is $(-2, -1)$

27. (a) : Given that

$$\tan(x - y) = 1 \Rightarrow \ x - y = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\text{Also } \sec(x + y) = \frac{2}{\sqrt{3}} \ \Rightarrow \ \cos(x + y) = \frac{\sqrt{3}}{2}$$

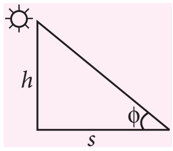
$$\Rightarrow x + y = \frac{\pi}{6}, \frac{11\pi}{6}$$

Now, $x + y > x - y$ (x, y are +ve)

$$\text{We take } x + y = \frac{11\pi}{6} \Rightarrow x - y = \frac{5\pi}{4}$$

$$\text{Solving we get } x = \frac{37\pi}{24}, \ y = \frac{7\pi}{24}.$$

28. (c) : $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$



$$\Rightarrow \ 16^{\sin^2\theta} + \frac{16}{16^{\sin^2\theta}} = 10$$

$$\Rightarrow \ x + \frac{16}{x} = 10 \ \ [\text{put } 16^{\sin^2\theta} = x]$$

$\Rightarrow \ x^2 - 10x + 16 = 0 \Rightarrow (x - 2)(x - 8) = 0$
$\Rightarrow \ x = 2, x = 8$
When $x = 2$, $16^{\sin^2\theta} = 2 \Rightarrow 4\sin^2\theta = 1$

$$\Rightarrow \ \sin^2\theta = \frac{1}{4} \Rightarrow \theta = \frac{\pi}{6} \text{ in } \left(0, \frac{\pi}{4}\right).$$

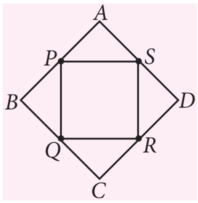
When $x = 8$, we do not get a solution in $\left(0, \frac{\pi}{4}\right)$

$$\therefore \ \alpha = \frac{\pi}{6}$$

$$\text{Now } s = h\cot\phi = \frac{h}{\sqrt{3}} \ \Rightarrow \ \cot\phi = \frac{1}{\sqrt{3}} \Rightarrow \phi = \frac{\pi}{3} = 2\alpha$$

29. (b) : $\cos^2 x(1 + \cos^2 x) = \cos^2 x + \cos^4 x$
$= \cos^2 x + (\cos^2 x)^2 = \cos^2 x + \sin^2 x = 1$

30. (c) : Let $ABCD$ is a quadrilateral. Joining the mid-pts of the sides of the quadrilateral, we get a parallelogram
We know diagonals of a parallelogram bisect each other



$$\therefore \ \left(\frac{0+4}{2}, \frac{1+6}{2}\right) = \left(\frac{3+a}{2}, \frac{5+b}{2}\right)$$

$\therefore \ (a, b) \equiv (1, 2)$

31. (d) : $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$

$$\Rightarrow \ f(x) = \begin{cases} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Since $f(x) = 0, \ \forall x \le 0$
$\therefore \ f(x)$ is a many-one function

$$\text{Let } y = \frac{e^x - e^{-x}}{e^x + e^{-x}}, x \ge 0 \Rightarrow \ \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{y}$$

$$\Rightarrow \ \frac{(e^x + e^{-x}) + (e^x - e^{-x})}{(e^x + e^{-x}) - (e^x - e^{-x})} = \frac{1+y}{1-y}$$

$$\Rightarrow \ \frac{e^x}{e^{-x}} = \frac{1+y}{1-y} \Rightarrow e^{2x} = \frac{1+y}{1-y}$$

$$\Rightarrow \; 2x = \log\left(\frac{1+y}{1-y}\right) \Rightarrow x = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) \qquad \text{...(i)}$$

Also $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x}-1}{e^{2x}+1}, \; x \ge 0$

Clearly $e^{2x} - 1 \ge 0 \;\; \forall \;\; x \ge 0$

$\therefore \;\; y \ge 0$ for $x \ge 0$ ...(ii)

From (i) and (ii), we get

$\therefore \;\;$ Range $f = [0, 1) \ne$ codomain f.

Hence f is not onto

Thus f is many one into mapping.

32. (a) : The function $\dfrac{1}{\sqrt{|f(x)| - f(x)}}$ will be defined, when $|f(x)| > f(x)$

$\Rightarrow \; f(x) < 0 \Rightarrow \log_e x + \log_x e < 0$

$\Rightarrow \; \dfrac{(\log_e x)^2 + 1}{\log_e x} < 0$

$\Rightarrow \; \log_e x < 0 \Rightarrow x < e^0 \Rightarrow x < 1 \Rightarrow 0 < x < 1$

33. (c) : $f(x) = x^3 + x^2 + 100x + 5\sin x$

$f'(x) = 3x^2 + 2x + 100 + 5\cos x$

$\quad = 3x^2 + 2x + 94 + (6 + 5\cos x) > 0$

$f(x)$ is an increasing function and consequently a one-one function.

Clearly $f(-\infty) = -\infty$, $f(\infty) = \infty$ and $f(x)$ is continuous, therefore range $f = R =$ codomain.

Hence f is onto

34. (c) : Reflexivity : Let (a, b) be an arbitrary element of $N \times N$. Then $(a, b) \in N \times N$

$\Rightarrow \; a, b \in N \Rightarrow a + b = b + a \Rightarrow (a, b)\; R(a, b)$

Thus $(a, b)\; R\; (a, b) \; \forall \; (a, b) \in N \times N$. So, R is reflexive on $N \times N$

Symmetry : Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\; R\; (c, d)$. Then

$(a, b)\; R\; (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a$

$\Rightarrow \; (c, d)\; R\; (a, b)$

Thus $(a, b)\; R\; (c, d) \Rightarrow (c, d)\; R\; (a, b) \forall (a, b), (c, d) \in N \times N$.

So R is symmetric on $N \times N$

Transitivity : Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\; R\; (c, d)$ and $(c, d)\; R(e, f)$.

Then, $(a, b)\; R\; (c, d) \Rightarrow a + d = b + c$

$\quad (c, d)\; R\; (e, f) \Rightarrow c + f = d + e$

$\therefore \;\; a + d + c + f = b + c + d + e$

$\Rightarrow \; a + f = b + e \Rightarrow (a, b)\; R\; (e, f)$

Thus, $(a, b)\; R(c, d)$ and $(c, d)\; R(e, f)$

$\Rightarrow \; (a, b)\; R\; (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$

So, R is transitive on $N \times N$

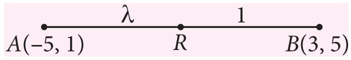
Hence, R being reflexive, symmetric and transitive, is an equivalance relation on $N \times N$

35. (c) : $2\tan 1095° + \cot 975° + \tan(-195°)$

$= 2\tan(12 \times 90° + 15°) + \cot(10 \times 90° + 75°) - \tan(2 \times 90° + 15°)$

$= 2\tan 15° + \cot 75° - \tan 15° = 2(2 - \sqrt{3}) = 4 - 2\sqrt{3}$

36. (a, c) : Given that R divides



AB in the ratio $\lambda : 1$

$\therefore \;\; R \equiv \left(\dfrac{3\lambda - 5}{\lambda + 1}, \dfrac{5\lambda + 1}{\lambda + 1}\right)$

Also given that, area of the ΔPQR is 2 sq. units

$\therefore \;\; \dfrac{1}{2}\left|1\left(2 - \dfrac{5\lambda+1}{\lambda+1}\right) + 7\left(\dfrac{5\lambda+1}{\lambda+1} - 5\right) + \dfrac{3\lambda-5}{\lambda+1}(5-2)\right| = 2$

$\Rightarrow \;\; \left|\dfrac{6\lambda - 42}{\lambda + 1}\right| = 4 \Rightarrow \dfrac{6\lambda - 42}{\lambda+1} = \pm 4 \Rightarrow \lambda = 23, \dfrac{19}{5}$

37. (a, c) : $\tan\theta = -\dfrac{1}{\sqrt{5}} \Rightarrow \theta$ may an angle of either 2^{nd} quadrant or 4^{th} quadrant.

$\therefore \;\; \cos\theta = \pm\dfrac{\sqrt{5}}{\sqrt{6}}$

38. (a, c) : $f(x) = 27x^3 + \dfrac{1}{x^3} = (3x)^3 + \left(\dfrac{1}{x}\right)^3$

$= \left(3x + \dfrac{1}{x}\right)^3 - 3 \cdot 3x \dfrac{1}{x}\left(3x + \dfrac{1}{x}\right) = 2^3 - 9 \times 2$

$f(x) = -10$

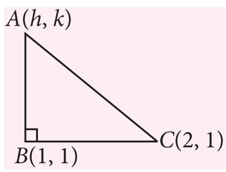
$\because \;\; \alpha, \beta$ are the roots of $3x + \dfrac{1}{x} = 2$

$\therefore \;\; f(\alpha) = f(\beta) = -10$

39. (a, b) : $f(x) = [x^2] + [x] + 1 - 3 = \{[x] + 2]\}\{[x] - 1\}$

So, $\forall \, x \in [-2, -1) \cup [1, 2) \Rightarrow f(x) = 0$

40. (b, d) : We know that,



$AB^2 + BC^2 = AC^2$

$\Rightarrow \;\; (h - 1)^2 + (k - 1)^2 + (2 - 1)^2 + (1 - 1)^2 = (h - 2)^2 + (k - 1)^2$

$\Rightarrow \;\; 2h = 2 \Rightarrow h = 1$

Also, $\dfrac{1}{2} BC \cdot AB = 1$

$\Rightarrow \;\; |1(k - 1)| = 2 \qquad [\because \; h = 1]$

$\Rightarrow \;\; k - 1 = \pm 2 \;\; \Rightarrow \;\; k = 3, -1$

◈◈

OLYMPIAD CORNER

1. (a) Suppose a_1, a_2 and a_3 are positive real numbers satisfying
$(a_1^2+a_2^2+a_3^2)^2 > 2(a_1^4+a_2^4+a_3^4)$.
Prove that a_1, a_2 and a_3 are the three sides of some triangle.

(b) Let n be an integer greater than 3. Suppose the inequality
$(a_1^2+a_2^2+.......+a_n^2)^2 > (n-1)(a_1^4+a_2^4+.......+a_n^4)$
holds for some positive real numbers $a_1, a_2, ..., a_n$. Prove that a_i, a_j and a_k are the three sides of some triangle for all i, j and k.

2. Let X be a finite set and $E(X)$ be the collection of subsets of X with an even number of elements. A real-valued function f is defined on $E(X)$ such that $f(D) > 1990$ for at least one D in $E(X)$ and $f(A \cup B) = f(A) + f(B) - 1990$ for all disjoint A and B in $E(X)$. Prove that X may be partitioned into two disjoint subsets P and Q such that $f(S) > 1990$ for all non-empty S in $E(P)$ and $f(T) \le 1990$ for all T in $E(Q)$.

3. Let n be a positive integer and a a positive real number. Determine the maximum value of $a^{k(1)}+a^{k(2)}+....+a^{k(s)}$, where s in an integer such that $1 \le s \le n$ and $k(1), k(2), ..., k(s)$ are positive integers with sum n.

4. Let a_0 and a_1 be integers. The sequence a_n is defined by $a_2 = 2a_1 - a_0 + 2$ and $a_{n+1} = 3a_n - 3a_{n-1} + a_{n-2}$ for $n \ge 2$. If for any positive integer m, the sequence contains m consecutive terms all of which are perfect squares, prove that every term of the sequence is a perfect square.

5. Let $x_1, x_2, ... , x_n$ be non-negative real numbers and let a be the minimum of these numbers.
(a) Prove that

$$\frac{1+x_1}{1+x_2}+\frac{1+x_2}{1+x_3}+...+\frac{1+x_{n-1}}{1+x_n}+\frac{1+x_n}{1+x_1}$$
$$\le n+\frac{(x_1-a)^2+(x_2-a)^2+...+(x_n-a)^2}{(1+a)^2}$$

(b) Prove that equality holds if an only if $x_1 = x_2 = ... = x_n$.

SOLUTIONS

1. **1st solution :** (a) The given inequality is equivalent to

$$0 < 2(a_1^2a_2^2+a_2^2a_3^2+a_3^2a_1^2)-(a_1^4+a_2^4+a_3^4)$$
$$=(a_1+a_2+a_3)(a_2+a_3-a_1)(a_3+a_1-a_2)(a_1+a_2-a_3).$$

We may assume that $a_1 \le a_2 \le a_3$. Hence the first three factors are positive, and it follows that so is the last one. Thus a_1, a_2 and a_3 satisfy the Triangle Inequality.

(b) We use induction on n. The case $n = 3$ is proved in (a). Suppose the result holds for some $n \ge 3$. For $n + 1$ real numbers $a_1, a_2, ... a_{n+1}$, the given inequality is equivalent to

$$\left(\sum_{k=1}^{n} a_k^2\right)^2 + 2a_{n+1}^2\sum_{k=1}^{n} a_k^2 + a_{n+1}^4 > n\sum_{k=1}^{n} a_k^4 + na_{n+1}^4,$$

or $$(n-1)a_{n+1}^4 - 2a_{n+1}^2\sum_{k=1}^{n} a_k^2 + n\sum_{k=1}^{n} a_k^4 - \left(\sum_{k=1}^{n} a_k^2\right)^2 < 0$$

Consider the left side as a quadratic polynomial in a_{n+1}^2. Since the leading coefficient is positive and the polynomial can assume negative values, it has real roots. Hence its discriminant must be positive. It follows that

$$4\left(\sum_{k=1}^{n} a_k^2\right)^2 - 4(n-1)\left(n\sum_{k=1}^{n} a_k^4 - \left(\sum_{k=1}^{n} a_k^2\right)^2\right) > 0,$$

or equivalently

$$\left(\sum_{k=1}^{n} a_k^2\right)^2 > (n-1)\sum_{k=1}^{n} a_k^4.$$

For $1 \le i < j < k \le n$, a_i, a_j and a_k satisfy the Triangle Inequality by the induction hypothesis. By symmetry, this also hold for $1 \le i < j < k \le n + 1$, completing the inductive argument.

2nd solution : (a) We use the same argument as in the First Solution.

(b) Since $n > 3$, it follows from Cauchy's Inequality that

$$(n-1)\sum_{k=1}^{n} a_k^4 < \left(\sum_{k=1}^{n} a_k^2\right)^2$$

$$= \left(\frac{a_1^2+a_2^2+a_3^2}{2} + \frac{a_1^2+a_2^2+a_3^2}{2} + a_4^2 + \dots\dots\dots + a_n^2\right)$$

$$\le (n-1)\left(\frac{(a_1^2+a_2^2+a_3^2)^2}{4} + \frac{(a_1^2+a_2^2+a_3^2)^2}{4} + a_4^4 + \dots\dots + a_n^4\right)$$

This is equivalent to $(a_1^2 + a_2^2 + a_3^2)^2 > 2(a_1^4 + a_2^4 + a_3^4)$, so that a_1, a_2 and a_3 satisfy the Triangle Inequality by (a). For $1 \le i < j < k \le n$. a_i, a_j and a_k also satisfy it by symmetry.

2. **1st solution :** Among the subsets in $E(X)$, choose P for which $f(P)$ is maximum. If there are more than one such choice, take one with the smallest number of elements. If there are several such subsets, take an arbitrary one. Define $Q = X - P$. Since $f(D) > 1990$ for some D in $E(X)$, $f(P) > 1990$. Let $S \ne P$ be a non-empty subset in $E(P)$. Then $P - S$ also belongs to $E(P)$, and has a smaller number of elements than P. Hence $f(P - S) < f(P)$. From
$f(P) = f(S \cup (P - S)) < f(S) + f(P - S) - 1990$,
$f(S) > 1990$. Now let T be any subset in $E(Q)$. Then $f(P \cup T) \le f(P)$. From $f(P \cup T) = f(P) + f(T) - 1990$,
$f(T) \le 1990$.

2nd solution : Define $g(M) = f(M) - 1990$ for all M in $E(x)$. Now $g(A \cup B) = g(A) + g(B)$ for disjoint A and B in $E(x)$. Let $X = \{a_1, a_2, \dots\dots, a_n\}$. For $1 \le i \le j \le n$, denote $g(\{a_i, a_j\})$ by $g_{i,j}$. Then we have
$g_{i,j} + g_{k,m} = g(\{a_i, a_j, a_k, a_m\}) = g_{i,k} + g_{j,m}$
For distinct integers i, j, k and m between 1 and n. We wish to define $g(\{a_i\}) = x_i$, for $1 \le i \le n$, so that $x_i + x_j = g_{i,j}$ for $1 \le i \le j \le n$. From $x_1 + x_2 = g_{1,2}$, $x_2 + x_3 = g_{2,3}$ and $x_1 + x_3 = g_{1,3}$, we have

$$x_1 = \frac{1}{2}(g_{1,2} + g_{1,3} - g_{2,3}),$$

$$x_2 = \frac{1}{2}(g_{1,2} + g_{2,3} - g_{1,3})$$

and $x_3 = \dfrac{1}{2}(g_{1,3} + g_{2,3} - g_{1,2})$

For $4 \le k \le n$, let

$$x_k = \frac{1}{2}(g_{1,k} + g_{2,k} - g_{1,2})$$

If k and m are distinct integers greater than 2, then

$$x_k + x_m = \frac{1}{2}(g_{1,k} + g_{2,m} + g_{1,m} + g_{2,k} - 2g_{1,2})$$

$$= \frac{1}{2}(g_{1,2} + g_{k,m} + g_{1,2} + g_{k,m} - 2g_{1,2})$$

$$= g_{k,m}$$

This relation also holds if k or m is 1 or 2, and the extension of the definition of g to all subsets of X is consistent. Now let $P = \{a \in X | g(a) > 0\}$ and $Q = X - P$. For any non-empty subset $\{a_1, a_2, \dots\dots, a_{2k}\}$ in $E(P)$,
$g(S) = g_{1,2} + g_{3,4} + \dots. + g_{2k-1,2k} > 0$,
which is equivalent to $f(S) > 1990$. We can show in the same way that $f(T) \le 1990$ for any T in $E(Q)$.

3. **1st solution :** If $0 < a \le 1$, a^l is non-increasing as the positive integer l increases. Hence we should take $s = n$ and $k(i) = 1$ for $1 \le i \le s$. The maximum value of $a^{k(1)} + a^{k(2)} + \dots. + a^{k(s)}$ ...(i)
is na. Now let $a > 1$. For all positive integers u and v,
$a(a^{u-1} - 1)(a^{v-1} - 1) \ge 0$,
or $\quad a^u + a^v \le a + a^{u+v-1}$
Hence we should take $k(i) = 1$ for $1 \le i \le s - 1$. The maximum value of (1) is then at most
$(s - 1)\, a + a^{n-(s-1)}$ ...(ii)
Note that $a + a^m = a^{m+1}$ if

$$m = \log_a \frac{a}{a-1} \qquad \text{...(iii)}$$

It follows that if

$$s \le (n+1) - \log_a \frac{a}{a-1}, \qquad \text{...(iv)}$$

The value of (ii) decreases as s decreases, so that we should take $s = n$ and obtain na. On the other hand, if

$$s > (n+1) - \log_a \frac{a}{a-1}, \qquad \text{...(v)}$$

the value of (ii) decreases as s increases, so that we should take $s = 1$ and obtain a^n. Hence the maximum value of (ii), and of (v), is $\max\{na, a^n\}$. For $n = 1$,

$na = a^n$. Suppose $n \geq 2$. Then $na = a^n$ when $a = n^{\frac{1}{n-1}}$.

Hence if $a \leq n^{\frac{1}{n-1}}$, the maximum value of (1) is na.

If $a > n^{\frac{1}{n-1}}$, the maximum value of (1) is a^n.

2nd solution : We prove by induction that the maximum value of (1) in the 1st solution is $\max\{na, a^n\}$. This is trivial for $n = 1$. Suppose it holds for some $n \geq 1$. Let

$k(1) + k(2) + ... + k(s) = n + 1$

Since $n + 1 - k(1) \leq n$, the induction hypothesis shows that

$a^{k(2)} + a^{k(3)} + ... + a^{k(s)} \leq \max\{(n + 1 - k(1))a, a^{n+1-k(1)}\}$.

The maximum value of (1) is at most

$\max\{a^{k(1)} + (n + 1 - k(1))a, a^{k(1)} + a^{n+1-k(1)}\}$

Now the functions $a^x + a(n + 1 - x)$ and $a^x + a^{n+1-x}$ are both convex. Hence their maximum values occur at the endpoints. It follows that

$a^{k(1)} + (n + 1 - k(1))a \leq \max\{(n + 1)a, a^{n+1}\}$

and $a^{k(1)} + a^{n+1-k(1)} \leq a + a^n \leq \max\{(n + 1)a, a^{n+1}\}$

The last inequality follows since

$a^{k(1)} + (n + 1 - k(1))a = a + a^n$ when $k(1) = n$. This completes the inductive argument. We can continue as in the 1st solution.

4. **1st solution :** Let $d_n = a_n - a_{n-1}$ for $n \geq 1$. Then

$0 = a_{n+1} - 3a_n + 3a_{n-1} - a_{n-2} = d_{n+1} - 2d_n + d_{n-1}$

For $n \geq 2$. Hence

$d_n - d_{n-1} = d_{n-1} - d_{n-2} = \ldots = d_2 - d_1 = a_2 - 2a_1 + a_0 = 2$

For $n \geq 1$, so that

$$a_n = a_0 + \sum_{k=1}^{n} d_k = a_0 + nd_1 + n(n-1)$$

$$= n^2 + (a_1 - a_0 - 1)n + a_0$$

By the hypothesis, there exists a positive integer t such that a_t and a_{t+2} are both squares. Hence $a_{t+2} - a_t \not\equiv 2 \pmod 4$. Since

$a_{t+2} - a_t = 4t + 4 + 2(a_1 - a_0 - 1)$,

We must have $a_1 - a_0 - 1 = 2\lambda$ for some integer λ and $a_n = (n + \lambda)^2 + a_0 - \lambda^2$

For $n \geq 0$. If $a_0 - \lambda^2 \neq 0$, let it have m divisors. Since the coefficient of the quadratic term in a_n is positive, there exists a positive integer n_0 such that $\{a_n\}$ is strictly increasing for $n > n_0$. By the hypothesis, there exists a positive integer $k > n_0$ such that for $1 \leq i \leq m$, $a_{k+i} = b_i^2$ for some positive integer b_i. We then have

$a_0 - \lambda^2 = b_i^2 - (k + i + \lambda)^2 = (b_i - k - i - \lambda)(b_i + k + i + \lambda)$

For $1 \leq i \leq m$, the integers $b_i + k + i\lambda$ are distinct since $b_i + i < b_{i+1} + i + 1$. Thus $a_0 - \lambda^2$ has at least $m + 1$ divisors, which is a contradiction. It follows that $a_0 - \lambda^2 = 0$ and $a_n = (n + \lambda)^2$ for $n \geq 0$.

2nd solution : As in the 1st Solution, we have $a_n = n^2 + \mu n + a_0$ for $n \geq 0$, where $\mu = a_1 - a_0 - 1$. Then

$$a_n = \left(n + \frac{\mu}{2}\right)^2 + a_0 - \frac{\mu^2}{4}$$

$$= \left(n + \frac{\mu + 1}{2}\right)^2 - n + a_0 - \frac{(\mu + 1)^2}{4}$$

$$= \left(n + \frac{\mu - 1}{2}\right)^2 + n + a_0 - \frac{(\mu - 1)^2}{4}$$

Let $n_0 > \max\left\{\frac{|\mu| + 1}{2}, \frac{(|\mu| + 1)^2}{4} + |a_0|\right\}$

When $n > n_0$, $\{a_n\}$ is strictly increasing, and we have

$$\left(n + \frac{\mu - 1}{2}\right)^2 < a_n < \left(\frac{n + \mu + 1}{2}\right)^2$$

From the hypothesis, $a_t = b^2$ for some integers $t > n_0$ and b. Hence

$$t + \frac{\mu - 1}{2} < b < t + \frac{\mu + 1}{2}$$

It follows that μ must be even, and

$$b = t + \frac{\mu}{2}$$

Hence $a_t = \left(t + \frac{\mu}{2}\right)^2$,

and it follows that $a_0 - \frac{\mu^2}{4} = 0$

We have $a_n = \left(n + \frac{\mu}{2}\right)^2$ for $n \geq 0$.

5. **1st solution**

(a) We use induction on n. For $n = 1$, both expressions are equal to 1. Suppose the desired inequality holds for some $n \geq 1$. Let a be the minimum of the non-negative real numbers $x_1, x_2, ... , x_{n+1}$. We may assume that x_{n+1} is the maximum. By the induction hypothesis, we have

MPP-2 CLASS XII ANSWER KEY

1.	(a)	**2.**	(b)	**3.**	(b)	**4.**	(d)	**5.**	(c)
6.	(b)	**7.**	(b, c)	**8.**	(a,b,c)	**9.**	(a, d)	**10.**	(b,c,d)
11.	(a, c)	**12.**	(c, d)	**13.**	(a, b, c, d)			**14.**	(c)
15.	(d)	**16.**	(b)	**17.**	(1)	**18.**	(4)	**19.**	(1)
20.	(9)								

$$\sum_{k=1}^{n-1}\frac{1+x_k}{1+x_{k+1}}+\frac{1+x_n}{1+x_1}\le n+\frac{1}{(1+a)^2}\sum_{k=1}^{n}(x_k-a)^2$$

It remains to show that

$$\frac{1+x_n}{1+x_{n+1}}+\frac{1+x_{n+1}}{1+x_1}-\frac{1+x_n}{1+x_1}\le 1+\left(\frac{x_{n+1}-a}{1+a}\right)^2$$

This is equivalent to

$$\frac{(x_{n+1}-x_n)(x_{n+1}-x_1)}{(1+x_{n+1})(1+x_1)}\le\frac{(x_{n+1}-a)}{(1+a)^2},$$

which holds since $a\le x_1$, $x_n\le x_{n+1}$.

(b) In order for equality to hold, we must have

$$x_1 = x_n = x_{n+1} = a$$

It follows that

$$x_1 = x_2 = \ldots = x_{n+1}.$$

2nd solution :

(a) Let $x_{n+1} = x_1$. Since

$$\sum_{k=1}^{n}\frac{x_k-x_{k+1}}{1+a}=0,$$

The desired inequality is equivalent to

$$\sum_{k=1}^{n}\frac{1+x_k}{1+x_{k+1}}\le\sum_{k=1}^{n}\left(1+\frac{x_k-x_{k+1}}{1+a}+\left(\frac{x_{k+1}-a}{1+a}\right)^2\right)$$

For $1\le k\le n$,

$$\frac{1+x_k}{1+x_{k+1}}\le 1+\frac{x_k-x_{k+1}}{1+a}+\left(\frac{x_{k+1}-a}{1+a}\right)^2$$

is equivalent to

$$\frac{(x_k-x_{k+1})(a-x_{k+1})}{(1+a)(1+x_{k+1})}\le\left(\frac{x_{k+1}-a}{1+a}\right)^2$$

If $x_k > x_{k+1}$, the left side is non-positive and the inequality obviously holds. It still holds if $x_k \le x_{k+1}$ since $a \le x_k$.

Hence the desired inequality follows.

(b) For equality to hold, we must have

$$\frac{x_{k+1}-x_k}{1+x_{k+1}}=\frac{x_{k+1}-a}{1+a}$$

For $1 \le k \le n$. This is equivalent to

$(x_{k+1} - a)^2 + (1 + a)(x_k - a) = 0$

Since both terms on the left side are non-negative, each is equal to 0, so that $x_1 = x_2 = \ldots = x_n = a$.

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. Find all possible triplets (x, y, z) such that $(x+y)+(y+2z)\cos 2\theta+(z-x)\sin^2\theta=0$, for all θ.

Niranjan Kumar, Haryana

Ans. We have,

$$(x+y)+(y+2z)\cos 2\theta+(z-x)\left(\frac{1-\cos 2\theta}{2}\right)=0$$

$$\Rightarrow (2x+2y)+(2y+4z)\cos 2\theta + (z-x)-(z-x)\cos 2\theta=0$$

$$\Rightarrow (2x+2y+z-x) + (2y+4z-z+x)\cos 2\theta=0$$

$$\Rightarrow (x+2y+z)+(x+2y+3z)\cos 2\theta=0 \text{ for all values of } \theta.$$

$$\Rightarrow x+2y+z=0 \text{ and } x+2y+3z=0$$

$$\therefore \frac{x}{6-2}=\frac{y}{1-3}=\frac{z}{2-2}=k$$

$$\Rightarrow x=4k,\ y=-2k,\ z=0 \text{ or } x=2k,\ y=-k,\ z=0.$$

i.e., $(2k, -k, 0)$ for any $k\in R$.

Hence, there are infinite number of triplets.

2. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\cdot {}^{3n}C_{2r-1}=0$ where $k=\frac{3n}{2}$ and n is an even positive integer.

Aniket Das Gupta, West Bengal

Ans. Given n is an even positive integer.

Let $n=2m$ $\therefore$ $k=3m,\ m\in N$

$$\text{L.H.S.}=\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1}=\sum_{r=1}^{3m}(-3)^{r-1}\,{}^{6m}C_{2r-1}$$

$$={}^{6m}C_1-3\cdot{}^{6m}C_3+3^2\,{}^{6m}C_5-\ldots+(-3)^{3m-1}\,{}^{6m}C_{6m-1} \quad \ldots\text{(i)}$$

Consider

$$(1+i\sqrt{3})^{6m}={}^{6m}C_0+{}^{6m}C_1(i\sqrt{3})+{}^{6m}C_2(i\sqrt{3})^2+{}^{6m}C_3(i\sqrt{3})^3+{}^{6m}C_4(i\sqrt{3})^4+{}^{6m}C_5(i\sqrt{3})^5+\ldots+{}^{6m}C_{6m-1}(i\sqrt{3})^{6m-1}+{}^{6m}C_{6m}(i\sqrt{3})^{6m}$$

or $$2^{6m}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{6m}$$

$$={}^{6m}C_0+{}^{6m}C_1(i\sqrt{3})+{}^{6m}C_2(i\sqrt{3})^2++{}^{6m}C_3(i\sqrt{3})^3+{}^{6m}C_4(i\sqrt{3})^4+{}^{6m}C_5(i\sqrt{3})^5+\ldots+{}^{6m}C_{6m-1}(i\sqrt{3})^{6m-1}+{}^{6m}C_{6m}(i\sqrt{3})^{6m}$$

or $2^{6m}[\cos 2\pi m+i\sin 2\pi\, m]$

$$=({}^{6m}C_0-3\,{}^{6m}C_2+3^2\cdot{}^{6m}C_4-\ldots+(-3)^{3m}\,{}^{6m}C_{6m})+i\sqrt{3}({}^{6m}C_1-3\,{}^{6m}C_3+3^2\,{}^{6m}C_5-\ldots+(-3)^{3m-1}\,{}^{6m}C_{6m-1})$$

Comparing the imaginary part on both sides we get

$$\sqrt{3}({}^{6m}C_1-3\cdot{}^{6m}C_3+3^2\,{}^{6m}C_5-\ldots+(-3)^{3m-1}\,{}^{6m}C_{6m-1})=0$$

or $${}^{6m}C_1-3\,{}^{6m}C_3+3^2\,{}^{6m}C_5-\ldots+(-3)^{3m-1}\,{}^{6m}C_{6m-1}=0$$

$$\Rightarrow \sum_{r=1}^{3m}(-3)^{r-1}\,{}^{6m}C_{2r-1}=0$$

$$\Rightarrow \sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1}=0 \quad (\text{where } n=2m)$$

3. Let $f(z)$ be a polynomial in z having complex coefficients. When $f(z)$ is divided by z^2+z+1 and $z-1$, remainders are $2\omega z+\omega$ and $5-\omega$ respectively $\left(\text{where } \omega=\frac{-1+i\sqrt{3}}{2}\right)$. Find the remainder, when $f(z)$ is divided by z^3-1.

Pooja Jain, Ahmedabad

Ans. Since $f(z)$ being divided by z^2+z+1 gives remainder $2\omega z+\omega$.

We get $f(z)=\alpha(z)\cdot(z^2+z+1)+2\omega z+\omega$, where $\alpha(z)$ is some polynomial.

Similarly, $f(z)=\beta(z)(z-1)+5-\omega$, where $\beta(z)$ is another polynomial.

Now, $f(\omega)=2\omega^2+\omega=\omega^2-1$ ...(i)

and $f(\omega^2)=2\omega^3+\omega=2+\omega$ ...(ii)

$(\because 1+\omega+\omega^2=0 \text{ and } \omega^3=1)$

and $f(1)=5-\omega$ ...(iii)

Now if $f(z)$ be divided by z^3-1, we get remainder of the from az^2+bz+c where a, b and c remains to be evaluated.

Now $f(z)=\phi(z)(z^3-1)+az^2+bz+c$

$f(\omega)=a\omega^2+b\omega+c=\omega^2-1$ [From (i)]

$f(\omega^2)=a\omega+b\omega^2+c=2+\omega$ [From (ii)

$f(1)=a+b+c=5-\omega$ [From (iii)]

Hence, from these equations, we get

$$a=\frac{2\omega^2-2\omega+7}{3},\ b=\frac{2\omega+5}{3},\ c=\frac{\omega^2+6}{3}$$

Hence, required remainder az^2+bz+c is given by

$$\frac{1}{3}[(2\omega^2-2\omega+7)z^2+(2\omega+5)z+\omega^2+6].$$

MATHS MUSING

SOLUTION SET-163

1. **(d):** $z = iy$, $y \in R$, $a = \cos\alpha + i\sin\alpha$

$\therefore$ $-(\cos\alpha + i\sin\alpha)y^2 + iy + 1 = 0$

$\Rightarrow$ $y^2\cos\alpha = 1$, $y\sin\alpha = 1$, $y \neq 0$

$\Rightarrow$ $\cos\alpha - \sin^2\alpha = 0$, $\cos^2\alpha + \cos\alpha - 1 = 0$

$$\cos\alpha = \frac{\sqrt{5}-1}{2}, \sec^2\alpha = \left(\frac{\sqrt{5}+1}{2}\right)^2$$

$$\tan^2\alpha = \left(\frac{\sqrt{5}+1}{2}\right)^2 - 1 = \frac{\sqrt{5}+1}{2} \Rightarrow \tan(\arg a) = \sqrt{\frac{\sqrt{5}+1}{2}}.$$

2. **(c) :** $\Sigma \tan A = \prod \tan A$

$$\tan A = k, \tan C = \frac{3}{k}, \tan B = \frac{k^2+3}{2k}$$

$$\therefore \quad \frac{\sin A \sin C}{\sin B} = \frac{3k}{k^2+3}$$

3. **(c) :** $a^{12}\, r^{66} = 8^{2014} = 2^{6042} \Rightarrow a^2\, r^{11} = 2^{1007}$

Let $a = 2^{\alpha}$, $r = 2^{\beta}$. Then, $2\alpha + 11\beta = 1007$

$\Rightarrow$ $(\alpha, \beta) = (498, 1), (487, 3), ..., (3, 91)$ *i.e.*, 46 pairs.

4. **(c) :** $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{r} = \dfrac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{a}} + \lambda\vec{a}$

$$\vec{r}\cdot\vec{c} = 3 \Rightarrow \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{\vec{a}\cdot\vec{a}} + \lambda\vec{a}\cdot\vec{c} = 3$$

$\therefore$ $3 = \lambda$, $\vec{r} = \hat{i} - \hat{k} + 3(\hat{i} + \hat{j} + \hat{k})$

$\therefore$ $\vec{r} = 4\hat{i} + 3\hat{j} + 2\hat{k}$, $|\vec{r}| = \sqrt{29}$

5. **(b):** $\dfrac{3}{1+x^3} = \dfrac{1}{x+1} - \dfrac{(2x-1-3)}{2(x^2-x+1)}$

$$\therefore \int_0^1 \frac{3dx}{1+x^3} = \ln\frac{x+1}{\sqrt{x^2-x+1}}\Bigg|_0^1 + \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\Bigg|_0^1$$

$$= \ln 2 + \sqrt{3}\frac{\pi}{3} = \ln 2 + \frac{\pi}{\sqrt{3}}$$

6. **(b, d):** The straight line with slope m, passing through $(2, -3)$ is $y + 3 = m(x - 2)$.

It is tangent to $(x + 1)^2 = 2y - 1$

Eliminating y, $x^2 + 2(1 - m)x + 4m + 8 = 0$

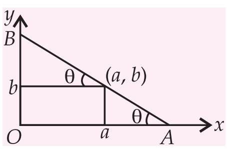
Its roots are equal.

$\therefore$ $(1 - m)^2 = 4m + 8 \Rightarrow m^2 - 6m - 7 = 0 \therefore m = -1, 7$

7. **(c) :** $OA = a + b\cot\theta$

$OB = b + a\tan\theta$

$(a, b) = (4, 1)$



$\Rightarrow$ $OA + OB = 5 + 4\tan\theta + \cot\theta$

$\geq 5 + 2\sqrt{4\tan\theta\cdot\cot\theta} = 9$ [By A.M. – G.M. inequality]

8. **(b):** $AB = a\sec\theta + b\operatorname{cosec}\theta = f(\theta)$

$$f'(\theta) = 0 \Rightarrow \frac{a}{\cos^3\theta} = \frac{b}{\sin^3\theta}$$

$$\therefore \quad \frac{a^{2/3}}{\cos^2\theta} = \frac{b^{2/3}}{\sin^2\theta} = \frac{a^{2/3}+b^{2/3}}{1}$$

$\therefore$ Minimum value of $AB = f(\theta) = (a^{2/3} + b^{2/3})^{3/2}$

But +$(a, b) = (64, 27)$

$\Rightarrow$ Minimum value of $AB = (16 + 9)^{3/2} = 125$

9. **(7):** $\sin x + \cos x = \dfrac{1}{4} - y$ where, $y = \sin x \cos x$

Squaring, $1 + 2y = \dfrac{1}{16} - \dfrac{y}{2} + y^2 \Rightarrow y = \dfrac{5}{4} - \dfrac{\sqrt{10}}{2}$

$\therefore$ $(1-\sin x)(1-\cos x) = \dfrac{13}{4} - \sqrt{10}$; $m = 13, n = 4, p = 10$

$\therefore$ $m + n - p = 7$

10. **(b): (P) $\to$ 3; (Q) $\to$ 1; (R) $\to$ 4; (S) $\to$ 2**

(P) Here, $(\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & -3 & 4 \\ 6 & 1 & -2 \end{vmatrix} = 2\hat{i} + 14\hat{j} + 13\hat{k}$

$$\therefore \quad d = \frac{|2\hat{i}+14\hat{j}+13\hat{k}|}{|6\hat{i}+\hat{j}-2\hat{k}|} = \sqrt{\frac{369}{41}} = 3$$

(Q) $z = \omega, \omega^2$

$$\therefore \quad \sum_{r=1}^{6}\left(\omega^r + \frac{1}{\omega^r}\right)^2 = 1+1+4+1+1+4 = 12$$

(R) Let $g(x) = f(x) - x \Rightarrow g(0) = g(1) = 0$

$f'(x) = 1 + g'(x)$

$$\therefore \quad \int_0^1 (f'(x))^2\,dx = \int_0^1 1\,dx + 2\int_0^1 g'(x)dx + \int_0^1 (g'(x))^2\,dx$$

$$= 1 + 2[g(x)]_0^1 + \int_0^1 (g'(x))^2\,dx = 1 + 0 + \int_0^1 (g'(x))^2\,dx \geq 1$$

(S) $t = \sin 2x + 2\cos^2 x \Rightarrow 1 - \sin 2x + 2\sin^2 x = 3 - t$

The equation becomes $3^t + \dfrac{27}{3^t} = 28 \Rightarrow 3^t = 1, 27$

$\therefore$ $t = 0, 3$

$\sin 2x + 2\cos^2 x < 3$

$\therefore$ $\sin 2x + 2\cos^2 x = 0 \Rightarrow \sin 2x + \cos 2x = -1$

$$\cos\left(2x - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$$

$$\therefore \quad 2x - \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4} \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$$